AA 601N : Astrophysical Fluids and Plasma

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Course Structure - I

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- **Review of Statistical Mechanics (Week 1)** : Concept of phase-space, Louisville Theorem, Distribution Function, Maxwell-Boltzmann Model, Fluid motion: Streamlines and Path-lines. Concept of fluid parcel.
- **Hydrodynamic Equations (Weeks 2 and 3)** : Eulerian and Lagrangian Framework Mass, Momentum and Energy conservation along Laws of Thermodynamics, Equation of state. Concept of Steady state, Effect of Gravity and Rotation, Virial Theorem, Centrifugal forces, Vortex flows. Viscous vs Inviscid flow, Bernoulli Equation.
- **Applications of HD Equations (Weeks 4 and 5)** : Accretion Disk, Hydro-static equilibrium and its application in star formation, Bondi Accretion and Parker Solar Wind. Concept of Shocks, Rankine-Hugoniot jump conditions.

MID SEMESTER EXAMINATION (MSE)

Course Structure - II

- **Shock physics (Week 6)** : Revision of Shocks : Adiabatic and Isothermal shocks, Application to Supernova Remnants and Jets.
- **Hydrodynamic Instabilities (Week 7)** : Concept of Linear perturbation theory, Kelvin Helmholtz Instability and Rayleigh Taylor Instability
- **Review of Plasma Physics (Weeks 8-9)** : Revision of Maxwell Equation, Plasma Properties, Motion of charged particle in EM field, Discharge physics.
- **Magneto-hydrodynamics (Week 10)** : Concept of Ideal MHD, Flux Freezing, Introducing MHD Conservation Equation

END SEMESTER EXAMINATION (MSE)

Standard References

1. Physics of Fluids and Plasmas by Arnab Ray Choudhari
2. Astrophysical Plasmas and Fluids by Vinod Krishan
3. Plasmas: The First State of Matter by Vinod Krishan
4. Principles of Astrophysical Fluid Dynamics by Cathie Clarke and Bob Carswell
5. An Introduction to Astrophysical Fluid Dynamics by Michael J Thompson

Marks Division

- Mid-Semester Examination : Weightage 20%
- End-Semester Examination : Weightage 40%
- Continuous Evaluation : Weightage 40%
  1. Quizzes
  2. Take Home Assignments
  3. Mini Numerical Projects

Astrophysical Fluids Dynamics : Area of Application.

Important Areas of applications include -

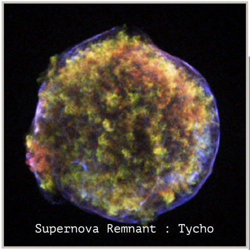
- Instabilities in astrophysical fluids
- Convection in stars
- Differential rotation and meridional flows in stars
- Stellar oscillations
- Astrophysical dynamos
- Magnetospheres of stars, planets and black holes
- Interacting binary stars and Roche-lobe overflow
- Tidal disruption and stellar collisions
- Supernovae
- Planetary Nebulae
- Jets and winds from stars and discs
- Star formation and the physics of the interstellar medium
- Astrophysical discs
- Other accretion flows (Bondi, Bondi–Hoyle, etc.)
- Processes related to planet formation and planet–disc interactions
- Planetary atmospheric dynamics
- Galaxy clusters and the physics of the intergalactic medium
- Cosmology and structure formation

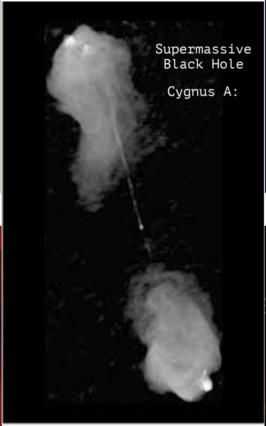
Astrophysical Fluids Dynamics : Area of Application.

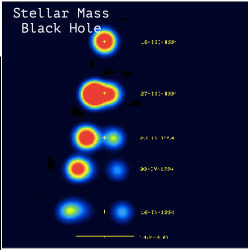
Astrophysical fluid dynamics (AFD) is a theory relevant to the description of the interiors of stars and planets, exterior phenomena such as discs, winds and jets, and also the interstellar medium, the intergalactic medium and cosmology itself.
A fluid description is not applicable -

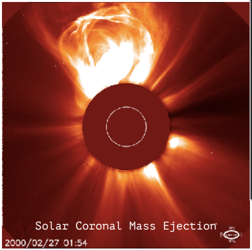
- in regions that are solidified, such as the rocky or icy cores of giant planets (under certain conditions)
- the crusts of neutron stars
- in very tenuous regions where the medium is not sufficiently collisional.

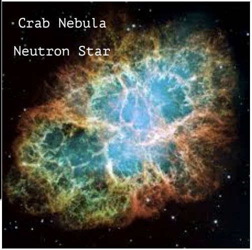
Astrophysical Flows











Partial Derivatives : Basics

- Chain Rule : Consider a function $f(x,y)$ and suppose x and y depend on another variable s i.e., $x(s)$, $y(s)$. Then

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

- Product Rule :

$$\frac{\partial uv}{\partial x} = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$$

- Commutation Rule

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Review of Statistical Mechanics

- Concept of Phase Space
- Concept of *Ensemble* and *Liouville* Theorem
- Introducing Equations in Phase space :
    1. Collision-less Boltzmann Equation
    2. Vlasov Equation.
    3. Collision Terms : Fokker Planck, Boltzmann Model
- Concept of Fluid and its description.

Phase Space

Consider a system of N particles. The time evolution of such a system is governed by the *Hamiliton's* equation for a given initial conditions.
The Hamiltonian H is a function of canonically conjugate variables : the generalized co-ordinates $\mathbf{q}_1$, $\mathbf{q}_2$, ... ,$\mathbf{q}_N$, corresponding momenta $\mathbf{p}_1$, $\mathbf{p}_2$, ... ,$\mathbf{p}_N$ and time t.
Hamilton's Equation are -

$$\frac{d\mathbf{q}_i}{dt} = \frac{\partial H}{\partial \mathbf{p}_i} \quad ; \frac{d\mathbf{p}_i}{dt} = -\frac{\partial H}{\partial \mathbf{q}_i}$$

Therefore, at any given time the system is completely defined if the Hamilton H and initial conditions are known.
Mechanical state of the system $\rightarrow$ single point in a 2N dimensional space.
Evolution of that single point $\rightarrow$ 2N vectors equations given above.
Such a 2N dimensional space made up of N generalized co-ordinates $\mathbf{q}_1$, $\mathbf{q}_2$, ..., $\mathbf{q}_N$ and N momenta $\mathbf{p}_1$, $\mathbf{p}_2$, ..., $\mathbf{p}_N$ is called the *Phase space*.

Function in Phase Space

Consider a function $f(\mathbf{q}, \mathbf{p}, t)$ of the 2N variables defined in phase space, then its derivative is given by

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial t} + \sum_{i=1}^{N} \frac{\partial f}{\partial \mathbf{q}_i} \cdot \frac{d\mathbf{q}_i}{dt} + \sum_{i=1}^{N} \frac{\partial f}{\partial \mathbf{p}_i} \cdot \frac{d\mathbf{p}_i}{dt} \\ &= \frac{\partial f}{\partial t} + [f, H] \end{aligned}$$

where [f, H] is called the Poisson bracket and its value is 0 if f is a constant of motion.

Exercise : For a system where Hamilton has no explicit time dependence, prove that the total energy of the system is conserved. What happens when the Hamilton does not have explicit dependence on say $\mathbf{q}_k$?

Gibb's Ensemble

A collection of *identical* systems that represent the same average properties is called an Gibb's Ensemble.

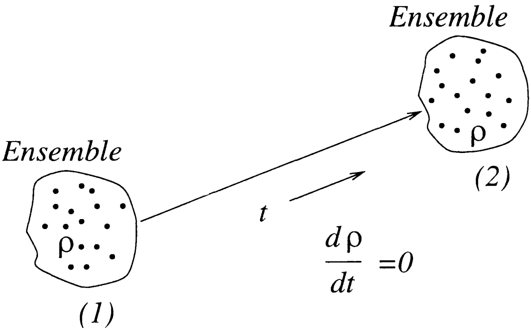
For example, in a harmonic oscillator the total energy is given as $p^2 + q^2$, and this we know remains invariant throughout the motion. Thus each pair (p, q) that preserves this in-variance is a member and a collection of such members forms an Ensemble.

Define density as the number of members in a volume $d\mathbf{q}_1 ... d\mathbf{q}_N d\mathbf{p}_1 ... d\mathbf{p}_N$ of the phase space at a given instant of time t.

$$\rho(\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_N, \mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_N, t) d\mathbf{q}_1 ... d\mathbf{q}_N d\mathbf{p}_1 ... d\mathbf{p}_N \tag{1}$$

Liouville Theorem

Liouville Theorem : *The density of states in an ensemble of many identical states with different initial conditions is constant along every trajectory in phase space*



What the Liouville Theorem does not mean ?

- Liouville's theorem does not imply that the density is uniform throughout phase space. In particular, if the Hamiltonian preserves energy, then one trajectory cannot visit two parts of phase space with different energy.
- Liouville's theorem does not imply that every point along a given path has the same density. In other words, suppose that two particles, A and B, follow the same trajectory, except that particle A leads particle B by a finite time (or equivalently, there is a finite distance in xp space between the two particles). Particle A could be in a region of different density than particle B.
- Liouville's theorem only holds in the limit that the particles are infinitely close together. Equivalently, Liouville's theorem does not hold for any ensemble that consists of a finite number of particles; instead the theorem describes the probability density in phase space of an ensemble consisting of an infinite number of possible states.

Liouville Equation

This equation governs the time evolution of density ρ in phase space -

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + [\rho, H]. \tag{2}$$

From the the Liouville's theorem we have L.H.S. $= 0$

$$\frac{\partial \rho}{\partial t} = -[\rho, H]. \tag{3}$$

Rather than solving set of canonical equations, we can determine the trajectory of system through just the above equation.

Exercise: Derive the Proof of the Liouville theorem.

Distribution Function

- The probability of finding the system in a volume element $d\mathbf{q}_1...d\mathbf{q}_N d\mathbf{p}_1...d\mathbf{p}_N$ is given by the function -

$$\rho(\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_N, \mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_N, t) d\mathbf{q}_1...d\mathbf{q}_N d\mathbf{p}_1...d\mathbf{p}_N \tag{4}$$

  The specific functional form of ρ in terms of constants of motion is known as the Distribution function.

- According to Liouville's theorem, we will have for a distribution of N particles -

$$\frac{d}{dt} f_N(\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_N, \mathbf{V}_1, \mathbf{V}_2, ..., \mathbf{V}_N, t) = 0 \tag{5}$$

- One particle distribution function - $f_1(\mathbf{q}_1, \mathbf{V}_1, t) d\mathbf{q}_1 d\mathbf{V}_1$ is the probability of finding one particle in a volume element $d\mathbf{q}_1 d\mathbf{V}_1$ of the phase space and is obtained by integration of f_N over all other co-ordinates $(\mathbf{q}_2, \mathbf{q}_3, ..., \mathbf{q}_N, \mathbf{V}_2, \mathbf{V}_3, ..., \mathbf{V}_N)$
- Similarly, a two particle distribution function - $f_2(\mathbf{q}_1, \mathbf{q}_2, \mathbf{V}_1, \mathbf{V}_2, t)$ can be defined. This represents joint probability of finding the two particles.

Collision-less Boltzmann Equation

The collision-less Boltzmann Equation can be derived using the Liouville's theorem and it can be shown that -

$$\frac{\partial f_1}{\partial t} + \mathbf{V}_1 \cdot \frac{\partial f_1}{\partial \mathbf{q}_1} + \frac{\mathbf{F}_{\text{ext}}}{m} \frac{\partial f_1}{\partial \mathbf{V}_1} = 0 \tag{6}$$

Several applications of this equation in Astrophysics -

- Study of Stellar motion in the Galaxy or star cluster. : $\mathbf{F}_{\text{ext}} = -m\nabla\phi_g$ Poisson Equation : $\nabla^2\phi_g = 4\pi G\rho_m(\mathbf{r}, t)$ where $\rho_m(\mathbf{r}, t) = m\int f(\mathbf{r}, \mathbf{V}, t)d\mathbf{V}$
- Collision-less plasma : The one between Sun and Earth – Space Weather The external force includes an additional term from the Lorentz Force. The Collision-less Boltzmann Equation for plasma in presence of electric field **E** and magnetic field **B** is called the Vlasov Equation.

Collision Term

In general, the Liouville's Equation for distribution function also has Collisional Terms due to presence of other particles/stars etc. especially in regions of high density.

- **Krook Collision Model** - RHS $= -\frac{1}{\tau}(f_1 - f_{\rm eq})$
  Question - Can you solve for f_1 assuming no spatial gradients and no-external force?
- **Boltzmann Collision Model** - Restricts the interaction among particles to only binary collisions. Applicable when - a) Particle density is low so that higher order interactions can be neglected. b) Particles experience only short range forces c) Within the range of forces, the short range force dominates over any external force d) The interactions are independent.

Concept of *Fluid Element*

Small size ...

The size of the fluid element, $l_{\rm fe}$, should be smaller than a scale length for change of any relevant fluid variable q -

$$l_{\rm fe} \ll \frac{q}{|\nabla q|} \tag{7}$$

... yet large enough ...

But at the same time it should be large enough to contain a sufficient number of particles so as to ignore noise due to finite number of particles (*discreteness noise*). Thus for a system with n as the number of particles per unit volume, we should have

$$n l_{\rm fe}^3 \gg 1 \tag{8}$$

... to be *collisional*!

The size of fluid element should be large enough so that the constituent particles *know* about local conditions through collisions -

$$l_{\rm fe} \gg \lambda = \frac{1}{n\sigma} \tag{9}$$

Validity of Fluid Approach

Collisions and Fluid Approach

The equations that govern the dynamics of fluids are essentially derived from micro-physical considerations. The essential idea is that if particles inside a fluid element interact with each other (not necessarily via physical collisions), then they will attain a distribution of particle speed that maximizes the entropy of the system at that temperature. This allows us to define fluid quantities like density, pressure and derive a relation between then in form of Equation of state.
In some cases, in-spite of frequent collisions (i.e., $t_{\rm coll} \ll T_{\rm scale}$), small deviations to the distribution function of particles can arise. These small deviations can be well accounted for by including appropriate non-ideal effects like viscosity, heat conduction, resistivity etc.

Fluid Approach Fails

Cases where the mean flight time of microscopic particles, $< \tau >$ is comparable to characteristic time scale i.e., $T_{\rm scale}$, the fluid approach is no longer valid. Alternatively, in astrophysical systems where the *mean free path*, $\lambda = \frac{1}{n\sigma}$ is comparable to characteristic length scale, $L_{\rm scale}$ of the system, the fluid equations can not be applied.

Validity of Fluid Approach : Exercise

Astrophysical System	$\rho,\ n$	T	$L_{\rm scale}$	$\lambda^{\dagger}$
Core of Sun-like star	$10^2 \mathrm{gcm}^{-3}$	$10^7\ K$	$\approx 0.05 R_{\odot}$	2×10^{-8} cm
Solar Corona	$10^{-15} \mathrm{gcm}^{-3}$	$10^6\ K$	$\sim 10 Mm$	?
ISM-Molecular clouds	$10^3 \mathrm{cm}^{-3}$	$10\ K$	80 pc	?
ISM-Ionized Medium	$10^{-3} \mathrm{cm}^{-3}$	$10^6\ K$	1000-3000 pc	~ 3 pc

† NOTE : The *Columb cross section* for collisions, $\sigma \approx 10^{-4} (T/K)^{-2}$ cm^2 and mean free path $\lambda = \frac{1}{n\sigma}$.

Multi-fluid, Hybrid, Kinetic Approach

In cases where the basic single-fluid approach fails, we can adopt more complicated *multi-fluid* or hybrid models which allows us to treat constituent particles separately. For example, in solar corona we can treat ions and electrons separately and study their dynamics along with interactions among them.
The most consistent approach is the Kinetic approach, which really solves the *Boltzmann Equation* from first principle, however they can not be applied to study very large systems.

Fluid Kinematics

Consider a fluid *parcel*, kinematics deals with the description of this parcel

- **Streamlines** Curves that are instantaneously tangent to the velocity vector of the flow. Lets say the $\mathbf{x}(s)$ is a streamline that depends on parameter s at one moment in time, then

$$\frac{d\mathbf{x}(s)}{ds} \times \mathbf{u}(s) = 0 \tag{10}$$

- **Pathlines** Trajectories of the fluid parcel given as -

$$\frac{d\mathbf{x}_p}{dt} = \mathbf{u}(\mathbf{x}_p(t), t) \tag{11}$$

Fluid Variables and Derivatives

Symbols and Meanings

Cartesian co-ordinate

$\mathbf{x} = x\hat{i} + y\hat{j} + z\hat{k}$ and time t.

Fluid Variable	Symbol
Velocity	$\mathbf{v}(\mathbf{x}, t)$
Density	$\rho(\mathbf{x}, t)$
Pressure	$P(\mathbf{x}, t)$
Magnetic Fields	$\mathbf{B}(\mathbf{x}, t)$
Specific Volume	$1/\rho$
Temperature	$\propto P/\rho$
Current Density	$\nabla \times \mathbf{B}$

Lagrangian v/s Eulerian

Eulerian viewpoint - Consider the variation of properties of the fluid at a fixed point in space. (i.e., attached to the inertial co-ordinate system), time derivative and any quantity $\mathcal{Q}$ is given by -

$$\frac{\partial \mathcal{Q}}{\partial t}$$

Lagrangian viewpoint - Consider the variation of properties of the fluid at a point that moves with the fluid at velocity $\mathbf{v}(\mathbf{x}, t)$, Lagranian time derivative of quantity $\mathcal{Q}$ is given by -

$$\frac{D\mathcal{Q}}{Dt} = \frac{\partial \mathcal{Q}}{\partial t} + \mathbf{v} \cdot \nabla \mathcal{Q}$$

More Notations and Vector Calculus

For any variable denoted by $\mathcal{Q}(\mathbf{x}, t) \equiv \mathcal{Q}(x, y, z, t)$, its partial derivatives are written as -

$$\mathcal{Q}_t \equiv \frac{\partial \mathcal{Q}}{\partial t}, \mathcal{Q}_x \equiv \frac{\partial \mathcal{Q}}{\partial x}, \mathcal{Q}_y \equiv \frac{\partial \mathcal{Q}}{\partial y}, \mathcal{Q}_z \equiv \frac{\partial \mathcal{Q}}{\partial z}$$

The *dot product* of two vectors $\mathbf{A} = (a_1, a_2, a_3)$ and $\mathbf{B} = (b_1, b_2, b_3)$ is given by -

$$\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Given a scalar quantity ϕ that depends on spatial co-ordinates x, y and z, the gradient operator ∇ as applied to scalar ϕ is a vector given by -

$$\mathrm{grad}\phi \equiv \nabla\phi \equiv (\phi_{\mathrm{x}}, \phi_{\mathrm{y}}, \phi_{\mathrm{z}}) \equiv \left(\frac{\partial \phi}{\partial \mathrm{x}}, \frac{\partial \phi}{\partial \mathrm{y}}, \frac{\partial \phi}{\partial \mathrm{z}}\right)$$

The *divergence operator* applies to any vector $\mathbf{A}$ results in a scalar quantity -

$$\mathrm{div}\,\mathbf{A} \equiv \nabla \cdot \mathbf{A} \equiv \frac{\partial \mathrm{a}_1}{\partial \mathrm{x}} + \frac{\partial \mathrm{a}_2}{\partial \mathrm{y}} + \frac{\partial \mathrm{a}_3}{\partial \mathrm{z}}$$

Conservation of mass

Equation of continuity applies to all systems that conserves mass.
If ρ is the density of system in some space then $\int \rho dV$ is the mass within volume V and it can change only due to the mass flux leaving across the surface bounding that volume i.e.,

$$\frac{\partial}{\partial t} \int \rho dV = - \int \rho \mathbf{v} \cdot \mathbf{dS} \tag{12}$$

Using the Gauss's Theorem we have

$$\int \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right] dV = 0 \tag{13}$$

implies,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \rightarrow \frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0 \tag{14}$$

Hence for in-compressible flows where density is constant, $\nabla \cdot \mathbf{v} = 0$.

Conservation of Momentum-I

Newton's second law of motion for a fluid element of mass $\rho\delta V$ can be expressed as -

$$\rho\delta V \frac{D\mathbf{v}}{Dt} = \delta\mathbf{F}_{\mathrm{body}} + \delta\mathbf{F}_{\mathrm{surface}} \tag{15}$$

where, $\delta\mathbf{F}_{\mathrm{body}} = \rho\delta V\mathbf{F}$ is a body force that acts on all points within the body, e.g., Gravity Force. where $\mathbf{F}$ is the body force per unit mass. The surface force $\delta F_{\mathrm{surface}}$ is the force acting on it across the surface bounding the fluid element. We can express the surface force in terms of the area element $d\mathbf{S}$ through a second-rank tensor P_{ij} as -

$$(d\mathbf{F}_{\mathrm{surface}})_i = -P_{ij}d\mathbf{S}_j \tag{16}$$

Implying the total force acting on the volume of the fluid element -

$$(\mathbf{F}_{\mathrm{surface}})_i = -\oint P_{ij}d\mathbf{S}_j = -\int \frac{\partial P_{ij}}{\partial x_j}\delta V \tag{17}$$

By convention, Pressure force is inward directed and area vector d$\mathbf{S}$ is outward.

Conservation of Momentum-II

Finally we can express the Newton's second law of motion for a fluid element as -

$$\rho \frac{Dv_i}{Dt} = \rho F_i - \frac{\partial P_{ij}}{\partial x_j} \tag{18}$$

For **ideal fluids**, we will assume here that force acting on an element of area inside or at the boundary is always perpendicular to the area element. $\rightarrow P_{ij} = P\delta_{ij}$

Surface force not perpendicular to the area element at the boundary is *shear force* $\rightarrow$ tangential stresses accounted by **viscosity**.

Therefore, we have conservation of momentum equation as -

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P + \mathbf{F} \tag{19}$$

Vorticity Equation

Define vorticity $\omega = \nabla \times \mathbf{v}$
Using the vector identity

$$\mathbf{v} \times (\nabla \times \mathbf{v}) = \frac{1}{2}\nabla(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{v} \cdot \nabla)\mathbf{v} \tag{20}$$

Using the above vector identity into the momentum conservation equation (by taking the curl) and assuming a conservative force, we have the *vorticity* equation.

$$\frac{\partial \omega}{\partial t} = \nabla \times (\mathbf{v} \times \omega) + \frac{1}{\rho^2}(\nabla \rho \times \nabla P) \tag{21}$$

What happens for a) incompressible fluid or b) barotropic fluid (where pressure and density have functional relation i.e., $P = P(\rho)$)

Vortex on Planetary Atmosphere

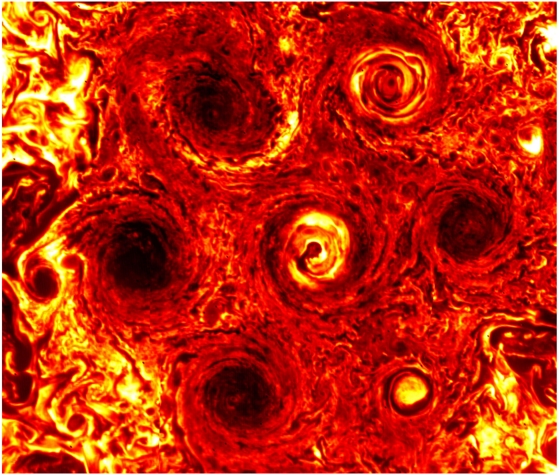


Figure: A new, smaller cyclone can be seen at the lower right of this infrared image of Jupiter's south pole taken on Nov. 4, 2019, during the 23rd science pass of the planet by NASA's Juno spacecraft. Credits: NASA/JPL-Caltech/SwRI/ASI/INAF/JIRAM

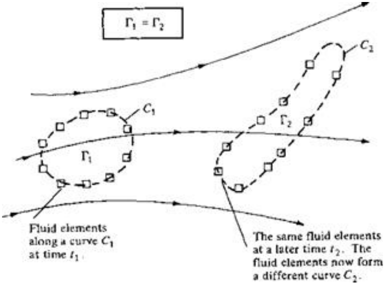
Kelvin's Vorticity Theorem

For incompressible and inviscid fluid, that satisfies the following vorticity equation

$$\frac{\partial \omega}{\partial t} = \nabla \times (\mathbf{v} \times \omega) \quad (22)$$

then the flux associated with vorticity is conserved, i.e.,

$$\frac{D}{Dt} \int_S \omega \cdot \mathbf{dS} = 0. \quad (23)$$



EXERCISE 1: Prove the theorem!

Conservation of Energy - I

Total energy :

$$E = \frac{1}{2}\rho \mathbf{v} \cdot \mathbf{v} + \rho\epsilon,$$

where ϵ is specific internal energy of the system.
Consider first the kinetic energy alone : Take dot product of the momentum conservation equation with $\mathbf{v}$,

$$\mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} = \frac{D}{Dt}\left(\frac{1}{2}\mathbf{v}^2\right) = -\frac{\mathbf{v} \cdot \nabla P}{\rho} + \mathbf{v} \cdot \mathbf{F} \tag{24}$$

This implies : Rate of change of specific kinetic energy is equal to the work done by the forces that act upon the fluid i.e., pressure gradient force and any external force.

Conservation of Energy - II

Now let us focus on the specific internal energy.
From *First Law of thermodynamics* we have,

$$d\epsilon = d\mathcal{Q} - PdV, \tag{25}$$

where $d\mathcal{Q} = \mathcal{H} - \Lambda$, i.e, amount of specific heat ($\mathcal{H}$) added to system minus the amount of cooling (Λ) per unit density.
Using definition of specific volume ($V = 1/\rho$), we have

$$\frac{D\epsilon}{Dt} = \frac{P}{\rho^2}\frac{D\rho}{Dt} + \frac{D\mathcal{Q}}{Dt} = -\frac{P}{\rho}(\nabla \cdot \mathbf{v}) + \frac{D\mathcal{Q}}{Dt} \tag{26}$$

Therefore,

$$\rho\frac{D}{Dt}\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + \epsilon\right) = -\nabla\cdot(P\mathbf{v}) + \rho\mathbf{v}\cdot\mathbf{F} + \rho\frac{D\mathcal{Q}}{Dt} \tag{27}$$

Gains and Losses in Energy

Let us define the rate of heat gain (or loss) as

$$-\mathcal{L} = \rho \frac{DQ}{Dt} \tag{28}$$

Assume that the heat flux in the system under-consideration is due to thermal conduction and therefore,

$$\mathcal{F} = -K\nabla T, \tag{29}$$

where negative sign implies that heat flows in direction opposite to temperature gradient and K is coefficient of thermal conductivity. Thus, the heat loss rate from a volume of fluid is equal to the heat flux integrated over the bounding surface-

$$\begin{aligned} \oint \mathcal{F} \cdot d\mathbf{S} &= \int \nabla \cdot \mathcal{F} dV \\ \mathcal{L} &= -\nabla \cdot (K\nabla T) \end{aligned} \tag{30}$$

Hydrodynamic Equations in Conservative Form

All the Hydrodynamics (HD) equations we have derived can be written a special form called the *conservative form* -

$$\frac{\partial m}{\partial t} + \nabla \cdot (F(m)) = 0 \tag{31}$$

where, m is any quantity and $F(m)$ is the flux associated with that quantity m

For example, say $m \equiv \rho$ then $F(m) \equiv \rho \mathbf{v}$ implies we have mass conservation equation -

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{32}$$

EXERCISE 2: Obtain the conservative form for momentum and energy conservation equation and thereby get the expression of their respective flux.

Hydrodynamic Equations in Conservative Form

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \\
\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v} + P) &= -\rho \nabla \phi \\
\frac{\partial (E_t + \rho\phi)}{\partial t} + \nabla \cdot (E_t + P + \rho\phi)\mathbf{v} &= 0
\end{aligned}
$$

where, $E_t = \rho\epsilon + \frac{1}{2}\rho v^2$ and gravity force per unit density $\mathbf{F}_g = -\nabla\phi$

Is the above set of equation complete ?? Assume $\phi \to 0$, we have 6 unknowns in general : ρ, P, $\mathbf{v} = (v_x, v_y, v_z)$ and ϵ, but we only have 5 equations - Mass conservation (1), energy conservation (1) and momentum conservation (3).

Equation of State

The relation that connects the density ρ, pressure P, temperature T or internal energy ϵ is called an Equation of state for the gas. In general, we can express EoS as $f(\rho, P, T) = 0$
For example ideal gas law :

$$PV = nRT \rightarrow PV - nRT = 0 \rightarrow f(\rho, V, T) = 0 \tag{33}$$

For *calorically perfect* gas, ideal gas law can be written as

$$P = \rho(\gamma - 1)\epsilon \tag{34}$$

where $\gamma = C_p/C_v$ is the ratio of specific heats and the specific internal energy $\epsilon = C_v T$.

Application – Hydrostatic Equilibrium

Consider a steady state 1D system along $\hat{\mathbf{z}}$ which is at rest ($\mathbf{v} = 0$). Obtain the density structure of this system assuming that the dominant mode of heat transfer is through conduction.

$$\nabla P = \rho \mathbf{F}_{\rm grav} \tag{35}$$

$$\nabla \cdot (K \nabla T) = 0 \tag{36}$$

Assuming that the pressure $P = P_0$ at z $= 0$ and $\mathbf{F}_{\rm grav} = -g\hat{\mathbf{z}}$, we have

$$P = P_0 - \rho g z \tag{37}$$

Considering a case of *isothermal ideal* gas, we have

$$\rho = \rho_0 \exp\left(-\frac{m_p g z}{k_B T}\right) \tag{38}$$

EXERCISE 3: Plot the variation of density and pressure upto 10 km above in the atmosphere of Mars. Assuming that radius of Mars is 3.38×10^8 cm and mass is 6.42×10^{23} kg and the density on the surface is 0.02 kg/m^3 and surface pressure is 6 mb and mean molecular weight of Martian atmosphere is 43.34

Application – Hydro-statics : Solar Corona

Solar corona is the hot and tenuous atmosphere just above the Sun ($T \sim 10^6$ K). Let us say we wish to study this system in *steady state* and as a static medium ($\mathbf{v} = 0$) We have in spherical geometry,

$$\frac{dP}{dr} = -\frac{GM_\odot}{r^2}\left(\frac{m_p P}{k_B T}\right) \tag{39}$$

$$\frac{d}{dr}\left(Kr^2 \frac{dT}{dr}\right) = 0 \tag{40}$$

Assume that $K \propto T^{5/2}$ and the boundary conditions as $T = T_0$ at $r = r_0$ (bottom part of corona) and $T = 0$ as $r \to \infty$. This gives pressure as

$$P = P_0 \exp\left(\frac{7GM_\odot m_p}{5k_B T_0 r_0}\left\{\left(\frac{r_0}{r}\right)^{5/7} - 1\right\}\right) \tag{41}$$

What is the value of P as $r \to \infty$??

EXERCISE 4: Plot the variation of pressure in terms of P_0 upto 1 AU above for Sun.

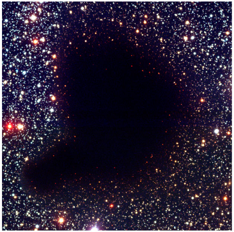
Thermodynamics Fun

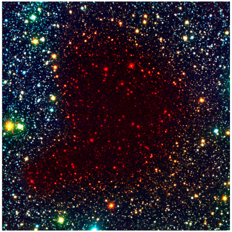
- Isobaric Process : Pressure of the system does not change i.e, P = constant
- Isothermal Process : Temperature does not change $\Delta T = 0$. For an ideal gas PV = constant (Boyle's Law)
- Adiabatic Process : Occurs without the transfer of heat between the system and surroundings $\Delta Q = 0$. For an ideal gas we have PV^{γ} = constant
- Isentropic Process : Thermodynamic process that is both adiabatic and reversible. Entropy is conserved in this process i.e., $\Delta s = 0$ or P/ρ^{γ} = constant

All the above processes can be described as a *Polytropic* process that satisfies the basic equation of PV^{Γ} = constant, where Γ is called the polytropic index. i.e., For an ideal gas : $\Gamma = 0$ (Isobaric), $\Gamma = 1$ (Isothermal), $\Gamma = \gamma$ (Adiabatic)

Bonner-Ebert Sphere - I

A hydrostatic equilibrium for a self gravitating spherically symmetric isothermal mass of an ideal gas. For example : Bok Globule (Image credit : ESO).


B, V, I


B, I, K

Can you estimate the density profile $\rho(r)$ variation with radius inside the sphere ?

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \tag{42}$$

Isothermal condition requires,

$$PV = \text{constant} \rightarrow P = \rho c_s^2 \quad \text{where} \quad c_s^2 = \frac{k_B T}{\mu m_H} \tag{43}$$

Maximum mass supported against self gravity by pressure due to isothermal

$$M_{\mathrm{BE}} \approx 1.18 \frac{c_{\mathrm{s}}^4}{G^{3/2} P_0^{1/2}}; \tag{44}$$

where $P_0 = \rho_0 c_{\mathrm{s}}^2$ is the central pressure

Bonner-Ebert Sphere - II

Let us define two non-dimensional quantities : $x = r/r_0$ and $\psi = \ln(\rho/\rho_0)$, where

$$r_0 = \frac{c_s}{\sqrt{4\pi G\rho_0}} \tag{45}$$

and ρ_0 is the density at the centre of the sphere. Then the above equation of hydrostatic balance transform to -

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\psi}{dx}\right) = -\exp(\psi) \tag{46}$$

The above second order equation can be written as two coupled first order equation

$$\frac{d\psi}{dx} = \frac{y}{x^2} \quad \text{and} \quad \frac{dy}{dx} = -x^2\exp(\psi) \tag{47}$$

The above equation is called the *Lane-Emden* equation and can be solved numerically using boundary conditions as $y = 0 \rightarrow \psi = 0$

Bernoulli's Theorem

Let us now assume that the fluid under consideration is **NOT** static i.e., $\mathbf{v} \neq 0$, but we are still in steady state. Assume a conservative body force $\mathbf{F} = -\nabla\phi$

Lets $\mathbf{dl}$ represents a line element vector along the streamline of the fluid. What would be the line element vector along the pathline? Further, from the definition of streamline what would be the value of $\mathbf{dl} \times \mathbf{v}$?

PROVE : For a steady state flow, the quantity

$$\mathcal{B} = \frac{1}{2}v^2 + \int \frac{dP}{\rho} + \phi$$

is constant along a streamline.

- Momentum conservation Equation in steady state
- Use the vector identity $\mathbf{v} \times (\nabla \times \mathbf{v}) = \nabla\left(\frac{1}{2}v^2\right) - (\mathbf{v}\cdot\nabla)\mathbf{v}$
- Line integral of the above equation along a streamline

Stream Function

Helmholtz Decomposition: A sufficiently smooth (continuously differentiable sufficient number of times) vector field can be resolved into the sum of an irrotational (curl-free) vector field and a solenoidal (divergence-free) vector field

$$\mathbf{v} = -\nabla\phi + \nabla \times \Psi \tag{48}$$

where, Ψ is called the stream function

- For an incompressible flow we can express : $\mathbf{v} = -\nabla \times \Psi$. Show that Ψ is a constant on streamlines
- For a two dimensional irrotational and incompressible flow, show that $\nabla\phi \cdot \nabla\Psi = 0$, where ϕ is the velocity potential.

From Ideal to Newtonian Fluids

Ideal Fluids

The force acting on the bounding surface element was normal and in opposite direction to the area vector. No shear force was taken into account.

In general, $\mathcal{P}_{\mathrm{ij}} = P\delta_{\mathrm{ij}} + \Pi_{\mathrm{ij}}$

Newtonian Fluids

The force due to shear is accounted for and for these fluids and the shear stress is directly proportional to velocity shear between the fluid layers. For example in 2D,

$$\Pi_{\mathrm{xy}} = -\mu \frac{dv_x}{dy},$$

where, μ is the coefficient of viscosity and the force due to shear acts in direction opposite to that of the shear.

Generalising the shear stress

In general, the expression of velocity gradient to obtain shear stress should be -

$$\frac{\partial v_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)}_{\text{pure shear}} + \underbrace{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)}_{\text{rotation}} \qquad (49)$$

EXERCISE : Show the the first term of pure shear say $\Lambda_{ij} = 0$ for case of rigid rotation with angular velocity Ω and velocity given as $\mathbf{v} = \Omega \times \mathbf{x}$ or $v_i = \epsilon_{ikl}\Omega_k x_l$

For Newtonian fluids, shear stress that depends linearly on velocity gradient. In general, any second rank tensor that linearly depends on symmetric combinations of velocity gradients is

$$\Pi_{ij} = a\underbrace{\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)}_{\text{pure shear}} + b\delta_{ij} \underbrace{\nabla \cdot \mathbf{v}}_{\text{trace of the tensor } \Lambda_{ij}} \qquad (50)$$

Navier Stokes Equation - I

Accounting for the fact that the pressure is isotropic and expressing $P = \frac{1}{3}\mathcal{P}_{ii}$, we should have the general form of shear stress to be **traceless**.

$$\Pi_{ij} = -\mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \nabla \cdot \mathbf{v} \right) \qquad (51)$$

One can demonstrate the same expression also from the microscopic perspective (using kinetic theory). So, the momentum conservation equation for Newtonian fluids become :

$$\rho \frac{Dv_i}{Dt} = \rho F_i - \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \nabla \cdot \mathbf{v} \right) \right] \qquad (52)$$

Navier Stokes Equation - II

Assuming that the coefficient of viscosity is not explicitly depend on space, we can write -

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{F} - \nabla P + \mu \left[\nabla^2 \mathbf{v} + \frac{1}{3} \nabla(\nabla \cdot \mathbf{v}) \right] \tag{53}$$

Further, neglecting an spatial variation of the compression (i.e., $\nabla \cdot \mathbf{v}$), we can write the Navier-Stokes Equation as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{F} - \frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{v} \tag{54}$$

where, $\nu = \mu/\rho$ is called the kinematic viscosity.

EXERCISE : What will be the corresponding Vorticity Equation?

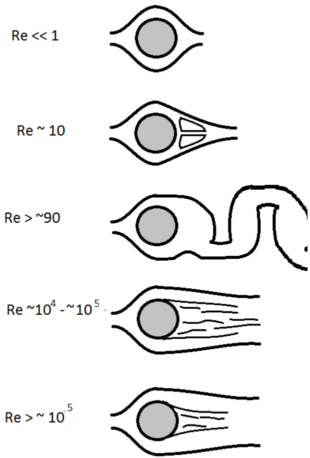
Reynolds Number

In general, the vorticity equation for incompressible viscous fluid can be written as :

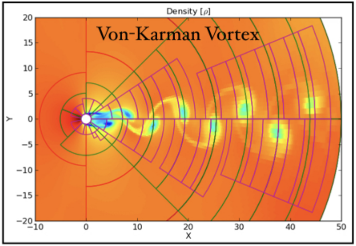
$$\frac{\partial \omega'}{\partial t'} = \nabla' \times (\mathbf{v}' \times \omega') + \frac{1}{\mathcal{R}e}\nabla'^2 \omega'$$

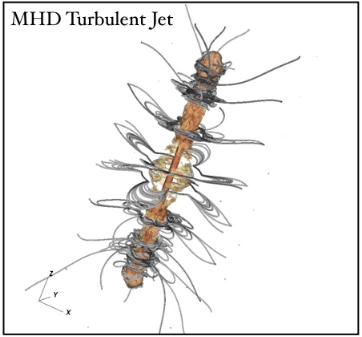
where, $\mathbf{x}' = \mathbf{x}L$, $\mathbf{v}' = \mathbf{v}V$, $t' = t(L/V)$, $\omega' = \omega(V/L)$ and the Reynolds number $\mathcal{R}e = LV/\nu$.

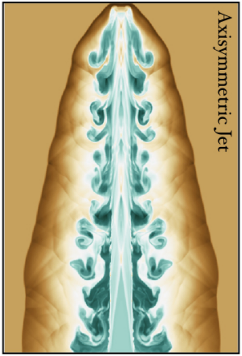
Typical values of Reynolds number in astrophysical flows $\sim 10^{12}$ (say for Sun)

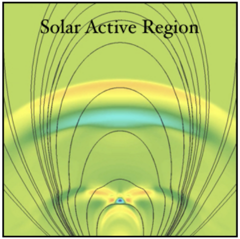


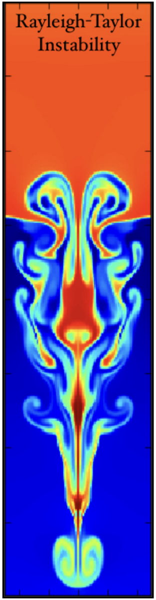
Turbulent Flows











Accretion Disks : Time Dependent Behaviour I

Several Assumptions :

- Consider thin accretion disk ($z \ll r$) in cylindrical geometry (r, ϕ, z) with the assumption of axi-symmetry (i.e., $\frac{\partial}{\partial \phi} = 0$)
- Assume v_ϕ dominant velocity component as compared to small accretion velocity v_r and $v_z = 0$. Also any variation of v_r and v_ϕ w.r.t z is neglected.
- In general, assume that coefficient of viscosity $\mu = \rho\nu$ is not a constant $\rightarrow$ True for accretion disk.
- Define disk surface density $\Sigma = \int \rho dz$. So, the angular momentum associated with the ring from r and $r + dr$ will be given by : $\Sigma(r) r^2 \Omega(r) 2\pi r dr$
- The velocity shear within the disk $\frac{dv_\phi}{dr} = \Omega + \boxed{r \frac{d\Omega}{dr}}$

Accretion Disks : Time Dependent Behaviour II

Combining from the mass and momentum conservation equation in cylindrical co-ordinates we can obtain the following evolution equation for angular momentum :

$$\frac{\partial(\Sigma r^2\Omega)}{\partial t} + \frac{1}{r}\frac{\partial(\Sigma r^3\Omega v_r)}{\partial r} = \mathcal{G} \tag{55}$$

where $\mathcal{G}$ is the term involving viscosity that we need to find out **OR** we define $G(r)$ as viscous torque and then have

$$\mathcal{G} = \frac{1}{2\pi r}\frac{\partial G}{\partial r} \tag{56}$$

and the viscous torque per unit area is obtained from the stress as

$$G(r) = \int r d\phi \int dz \mu r^2 \frac{d\Omega}{dr} = 2\pi\nu\Sigma r^3 \frac{d\Omega}{dr} \tag{57}$$

Accretion Disks : Time Dependent Behaviour III

Collecting all together we have :

$$\frac{\partial(\Sigma r^2\Omega)}{\partial t} + \frac{1}{r}\frac{\partial(\Sigma r^3\Omega v_r)}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(\nu\Sigma r^3\frac{d\Omega}{dr}\right) \tag{58}$$

EXERCISE : Show that the above equation for a Keplerian rotating disk i.e., $\Omega = \left(\frac{GM}{r^3}\right)^{1/2}$ can be simplified to

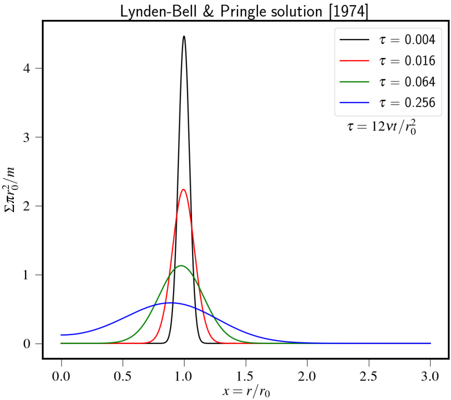
$$\frac{\partial\Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[\sqrt{r}\frac{\partial(\nu\Sigma\sqrt{r})}{\partial r}\right] \tag{59}$$

Accretion Disks : Time Dependent Behaviour Solution

For the simple case of Keplerian rotation and assuming that we have a dirac δ distribution of matter peaking at r_0 at time t $= 0$ (i.e., $\Sigma(r, t = 0) = m(2\pi r)^{-1}\delta(r - r_0)$) with constant kinematic viscosity *nu*.
The solution is

$$\Sigma(x, \tau) = \frac{m}{\pi r_0^2 \tau x^{1/4}} \exp\left(-\frac{1 + x^2}{\tau}\right) I_{1/4}\left(\frac{2x}{\tau}\right)$$

where $I_{1/4}$ is Modified Bessel Function.



Accretion disks : Steady State

- Meaning of steady state ?
- Gravity Force assuming a thin disk ($z \ll r$) ?
- Momentum conservation equation - radial component ?, vertical component ?
- Prove that gradient of pressure term is negligible in comparison to gravity
- Using Σ, get steady state mass and momentum conservation neglecting pressure gradient.
- Show that

$$\nu\Sigma = \frac{\dot{M}}{3\pi}\left[1 - \left(\frac{r_*}{r}\right)^{1/2}\right] \tag{60}$$

Accretion Disk : Energetics

EXERCISE : Derive the energy conservation equation for a Newtonian fluid and show that viscous dissipation rate per unit volume within the accretion disk is $\mu r^2 (\frac{d\Omega}{dr})^2$.

Therefore, the amount of energy per unit volume radiated away from the accretion disk is -

$$-\frac{dE}{dt} = \int \mu r^2 \left(\frac{d\Omega}{dr}\right)^2 dz = \nu \Sigma r^2 \left(\frac{d\Omega}{dr}\right)^2 \tag{61}$$

Expressing the above equation in terms of mass accretion rate $\dot{M}$ and integrating over the surface area of the accretion disk we get the **Accretion disk Luminosity**

$$L_{\mathrm{disk}} = \int_{r_*}^{\infty} \left(-\frac{dE}{dt}\right) 2\pi r dr = \frac{GM\dot{M}}{2r_*} \tag{62}$$

Perturbing the Hydrodynamic Equations

- Assume a homogenous ideal gas with density ρ_0 and pressure P_0 and in absence of any external force at rest $\mathbf{v}_0 = 0$.
- Suppose the pressure of this gas is perturbed i.e., $P_0 + P_1(\mathbf{x}, t)$ that gives rise to corresponding perturbation in density as $\rho_0 + \rho_1(\mathbf{x}, t)$
- The equation for mass conservation now becomes -

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot [(\rho_0 + \rho_1)\mathbf{v}_1] = 0 \tag{63}$$

  where, $\mathbf{v}_1(\mathbf{x}, t)$ is the velocity perturbation.
- Concept of Linear Perturbation Theory : Assume that the perturbation in density and pressure are much small as compared to its homogenous values and only the first order terms of perturbed quantities need to be considered i.e.,$\rho_1 \mathbf{v}_1 \approx 0$

Linear Perturbation in Ideal Fluids

- Linearize the mass and momentum conservation equation assuming an ideal fluid in absence of any external force $\mathbf{F} = 0$.
- The perturbed pressure is related to perturbed density in manner : $P_1 = a^2 \rho_1$, where the quantity a is

$$a = \sqrt{\frac{dP}{d\rho}} \tag{64}$$

- Using the above one can show that the final expression showing the evolution of density perturbation is :

$$\left(\frac{\partial^2}{\partial t^2} - a^2 \nabla^2 \right) \rho_1 = 0 \tag{65}$$

We get a **wave equation** with wave speed a.

Wave Equation Fourier Analysis

- For linear perturbations $\rightarrow$ Superposition principle holds true, so,

$$\rho_1(\mathbf{x}, t) = \rho_{1,0} \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)] \tag{66}$$

- Substitute in the above wave equation we can get a simple algebraic equation -

$$\omega^2 = a^2 k^2 \tag{67}$$

  The above relation is called the **dispersion relation**
- For the above case we have group and phase velocity $v_g = v_p = a$. Such waves are called *non-dispersive* waves. How will this change if we include the external force due to gravity $\mathbf{F} \neq 0$.
- Also the direction of perturbed velocity $\mathbf{v}_1$ is same as $\mathbf{k}$ implying that wave waves are *longitudinal*

Jeans Instability

Gas is Compressed $\rightarrow$ Excess pressure tried to smoothen the gas $\rightarrow$ gives rise to acoustic waves.

If Gravity is involved $\rightarrow$ The compressed region will try to pull more material towards itself $\rightarrow$ Quite insignificant in propagation of acoustic waves

Jeans Instability $\rightarrow$ If the self-gravity and enhanced gravitation in the region of compression overpowers the smoothening caused by pressure in the compressed region.

Perturbation in Gravitation potential $\Phi_0 + \Phi_1$, where the unperturbed potential should satisfy

$$\nabla^2 \Phi_0 = 4\pi G \rho_0 \tag{68}$$

and hydro-static balance gives

$$\nabla P_0 = -\rho_0 \nabla \Phi_0 \tag{69}$$

PROBLEM?? Uniform infinite gas can not satisfy the above set of equations! Ideally, one need to solve the above equations and then perturb the system, but, we follow the linear perturbation approach historically used by Jeans $\rightarrow$ *Jeans Swindle*

Jeans Swindle

Linear perturbation equations for mass, momentum conservation and the corresponding Poisson equation are :

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0 \tag{70}$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + c_s^2 \nabla \rho_1 = -\rho_0 \nabla \Phi_1 \tag{71}$$

$$\nabla^2 \Phi_1 - 4\pi G \rho_1 = 0 \tag{72}$$

Define all perturbed quantities in form of $\exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$ and get the *dispersion* relation as

$$\omega^2 = c_s^2 (k^2 - k_J^2) \tag{73}$$

where

$$k_J^2 = \frac{4\pi G \rho_0}{c_s^2} \tag{74}$$

Jeans Mass, Length and Free fall time

Jeans Length can be defined as $\lambda_J = 2\pi/k_J$ and Jeans Mass $M_J = (4/3)\pi\rho_0\lambda_J^3$.

- $k > k_J \rightarrow \omega^2 > 0$ (Density waves) What is the phase and group velocity?
- $k < k_J \rightarrow \omega^2 < 0$ (Jeans Instablity) Results in collapse with time scale τ_{ff}
- Time scale of Jeans collapse $\tau_g = 2\pi\left(4\pi G\rho_0 - k^2c_s^2\right)^{-1/2}$ which for the fastest growth $k \rightarrow 0$ gives the free fall time scale. i.e., $\tau_{ff} \approx \sqrt{\frac{1}{G\rho_0}}$

Estimating Jeans Quantities

Obtain values of Jeans Mass, Length and free fall time for the following cases :

- Molecular cloud : $T = 150\,K, n_0 = 10^8 cm^{-3}$ and have typical mass $M = 10 - 1000 M_\odot$
- Cosmological scales at decoupling of matter and radiation : $\rho_0 = 2M_\odot pc^{-3}, T = 3000K$.
- Diffuse HI cloud : $T = 50K, n_0 = 500cm^{-3}$ and have typical mass $M = 1 - 100 M_\odot$

Role of non-linear terms

- Consider the situation when the perturbed quantities are not very small and it is not possible to neglect the second order terms in perturbation.
- Then $\mathbf{v}_1 \cdot \nabla \mathbf{v}_1$ in the momentum equation would be one of the non-linear term in the momentum conservation equation :BIGGEST CULPRIT
- Consider the momentum conservation equation in 1D

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x} \tag{75}$$

- To understand role of the non-linear term just consider the RHS of above equation to be 0.

Burger's Equation and Characteristics

The equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{\partial v}{\partial t} + \frac{Dx}{Dt}\frac{\partial v}{\partial x} = \frac{Dv}{Dt} = 0 \tag{76}$$

is called the Burger's equation. We solve this equation using *method of characteristics*

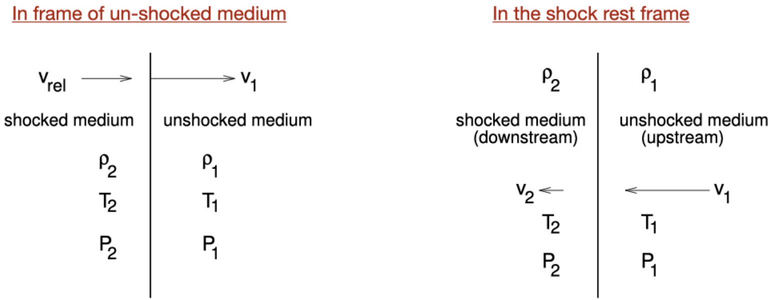
Characteristics are defined as lines on the $x - t$ plane. For the above equation, we know that these lines will be straight on the $x - t$ plane as they are the curves that represent

$$\frac{Dx}{Dt} = v = \text{constant}$$

Check for solution and more details about the method of characteristics for Burger's Equation : `http://www.clawpack.org/riemann_book/html/Burgers.html`

Structure of Shock waves

In the rest frame of shock, it will divide the region into *upstream* [un-disturbed] and *downstream disturbed* zones.



AIM: Find the relation between the upstream and downstream quantities. Basic shock physics : $\rho_2 > \rho_1$, $P_2 > P_1$, $T_2 > T_1$, $v_1 > v_2$

Rankine-Hugoniot Jump conditions

Under the steady state assumption we have all the flux (mass, momentum and energy flux) conserved across the shock i.e.,

$$\begin{aligned} \rho_1 v_1 &= \rho_2 v_2 \\ P_1 + \rho_1 v_1^2 &= P_2 + \rho_2 v_2^2 \\ \frac{1}{2} v_1^2 + \frac{\gamma P_1}{(\gamma - 1)\rho_1} &= \frac{1}{2} v_2^2 + \frac{\gamma P_2}{(\gamma - 1)\rho_2} \end{aligned} \tag{77}$$

3 equations, 6 unknowns $\rightarrow$ eliminate P_2 and v_2 and we get

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\gamma + 1)\mathcal{M}^2}{2 + (\gamma - 1)\mathcal{M}^2} = R \tag{78}$$

where $\mathcal{M} = \frac{v_1}{\sqrt{\gamma P_1/\rho_1}} = \frac{v_1}{c_{s,1}}$.

In the limit of strong shocks : $\mathcal{M} \gg 1$, we have $\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1}$

Rankine-Hugoniot Jump conditions

Shock jump conditions with adiabatic index γ and mach number $\mathcal{M}$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)}{(2/\mathcal{M}^2 + (\gamma-1)}$$
$$\frac{v_1}{v_2} = \frac{(\gamma+1)}{(2/\mathcal{M}^2 + (\gamma-1)}$$
$$\frac{P_2}{P_1} = \frac{2\gamma\mathcal{M}^2 - (\gamma-1)}{\gamma+1}$$
$$\frac{T_2}{T_1} = \frac{P_2}{\rho_2}\frac{\rho_1}{P_1}$$

In the case of strong shock limit i.e., $\mathcal{M} \gg 1$

$$\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1} \to 4$$
$$\frac{v_1}{v_2} = \frac{\gamma+1}{\gamma-1} \to 4$$
$$\frac{P_2}{P_1} = \frac{2\gamma\mathcal{M}^2}{\gamma+1} \to \frac{3}{4}\frac{\rho_1}{P_1}v_1^2$$
$$\frac{T_2}{T_1} = \frac{2\gamma(\gamma-1)\mathcal{M}^2}{(\gamma+1)^2} \to \frac{3}{16}\frac{\mu m_H}{k_B}v_1^2$$

EXERCISE : What happens for the case of Isothermal gas?

One dimensional gas flow : Extragalactic Jets

Consider a steady, adiabatic flow with velocity v(x) going through a pipe whose area in general varies with distance A(x)

$$\begin{aligned} \rho(x)v(x)A(x) &= \text{constant} \\ \frac{1}{\rho}\frac{d\rho}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{A}\frac{dA}{dx} &= 0 \end{aligned}$$

and the Euler equation is given as

$$v\frac{dv}{dx} = -\frac{c_s^2}{\rho}\frac{d\rho}{dx} \tag{79}$$

This implies

$$(1-\mathcal{M}^2)\frac{1}{v}\frac{dv}{dx} = -\frac{1}{A}\frac{dA}{dx} \tag{80}$$

De Laval Nozzle : Twin Exhaust Model