Relativistic Kinematics

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I. WHY HIGH ENERGIES?

In Particle Physics, we deal with elementary constituents of matter. By elementary we mean the particles have no substructure or they are point-like objects. However, the elementariness depends on the spatial resolution of the probe used to investigate the possible structure/sub-structure [1]. The resolution is Δr if two points in an object can just be resolved as separate when they are a distance Δr apart. Assuming that the probing beams themselves consist of point-like particles like electrons or positrons, the resolution is limited by the de Broglie wavelength of these beam particles, which is given by $\boxed{\lambda \ = \ h/p}$, where p is the beam momentum and h is Planck's constant. Hence beams of high momentum have short de Broglie wavelengths and can have high resolution. For example if we need to probe a dimension of $1 \ Fermi$ ($10^{-15} \ meter$) (let's say the inner structure of proton, the charge radius of proton being $\sim 0.840 \ fm$), we need to use a beam of momentum $1.47 \ GeV$ which is given by de Broglie's above expression. Figure 1 shows a schematic picture where a low energy probe fails to probe the inner structure of an object unlike a high energy probe with higher resolution. In addition to the above consideration, Einstein's formula $E \ = \ mc^2$ helps us to produce particles of higher masses (like the massive gauge bosons, Higgs etc.) in nature's way.
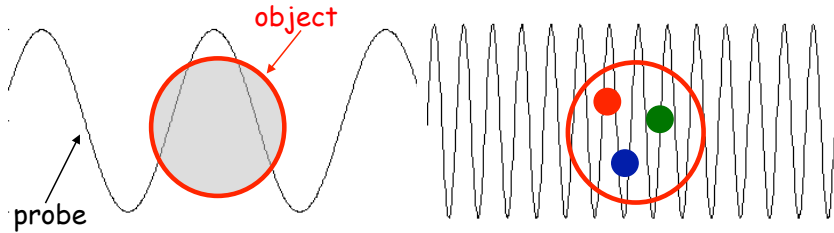

FIG. 1: Left: A low-energy probe probing an object, Right: a high-energy probe able to probe deep inside of the object because of higher resolution.

II. SPECIAL THEORY OF RELATIVITY & INVARIANTS

In particle physics, the particles are treated relativistically, meaning $E \approx pc \gg mc^2$ and thus special theory of relativity becomes an mathematical tool in describing the particle kinematics.

- Space and time can not be treated independently as is done in Newtonian mechanics.

- Physical objects that were treated as an independent three component vector and a scalar in non-relativistic physics mix in high-energy phenomena.

- Combined to form a four-component Lorentz vector that transforms like a time and space coordinate.

For their consistent and unified treatment, one relies on Einstein's theory of special relativity (STR), having the following two underlying principles.

- <u>Invariance of velocity of light</u>: velocity of light always remains as the constant c in any *inertial frame.*

- <u>Relativity Principle</u>: This requires covariance of the equations, namely the physical law should keep its form invariant in any inertial frame of reference. In mathematical language this amounts to the fact that physical laws have to be expressed in Lorentz tensors.

Note that the principle of relativity applies to Galilei transformation and is valid in Newtonian mechanics as well. But the invariance of the velocity of light necessitates Lorentz transformation in changing from one inertial system to another that are moving relative to each other with constant speed.

A. Lorentz Transformation

Consider a Lorentz boost in x-direction. Here, a particle at $(t, x, y, z,)$ in a coordinate frame L is boosted to (t', x', y', z') with velocity v. This statement is equivalent to changing to another coordinate frame L' which is moving in the x-direction at velocity $-v$. L' is assumed to coincide with L at $t = t' = 0$. Then the two coordinates are related by the following equations:

$$
\begin{aligned}
t \to t' &= \frac{t + (v/c^2)\, x}{\sqrt{1 - (v/c)^2}} \Rightarrow x^{0\prime} = \ \gamma\left(x^0 + \beta x\right), \\
x \to x' &= \frac{x + vt}{\sqrt{1 - (v/c)^2}} \Rightarrow x^{1\prime} = \ \gamma\left(\beta x^0 + x\right), \\
x^{2\prime} &= \ x^2, \\
x^{3\prime} &= \ x^3
\end{aligned}
\tag{1}
$$

where, $\beta = v/c$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. The above equations can be written in matrix form as:

$$
\begin{bmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{bmatrix} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}
\tag{2}
$$

1. The Proper Time (τ)

It is the time an observer feels in the observer's rest frame.

Proper time $\boxed{d\tau \equiv dt\sqrt{1 - \beta^2}}$ is a Lorentz invariant scalar.
<u>Proof:</u>

$$
\begin{aligned}
ds^2 &= (cdt)^2 - dx^2 - dy^2 - dz^2 \\
&= c^2 dt^2 \left[1 - \frac{dx}{dt}^2 - \frac{dy}{dt}^2 - \frac{dz}{dt}^2 \right] \\
&= c^2 dt^2 \left(1 - \beta^2 \right) \\
&= (cd\tau)^2
\end{aligned}
$$

is Lorentz invariant by definition.

B.   What is need of the variable called "Rapidity"?

Successive Lorentz boost in the same direction is represented by a single boost, where the transformation velocity is given by

$\beta'' = |v/c|'' = \frac{\beta + \beta'}{1 + \beta\beta'}$

Proof:
Assume velocity v' in frame L is observed as v'' in frame L'', where the frame L' is travelling in the x-direction with $-v$ in frame L. The coordinates $(t', x^{1'})$ are expressed in terms of (t, x^1) using the usual Lorentz transformation equations given by Eqn. 1. Omitting other coordinates for simplicity, one obtains:

$$
\begin{aligned}
x^{0'} &= \gamma \left(x^0 + \beta x^1 \right) \\
x^{1'} &= \gamma \left(\beta x^0 + x^1 \right) \\
\beta' &= \frac{v'}{c} = \frac{dx^1}{dx^0}
\end{aligned}
$$

then

$$
\begin{aligned}
\beta'' = \frac{dx^{1'}}{dx^{0'}} &= \frac{\gamma \left(\beta dx^0 + dx^1 \right)}{\gamma \left(dx^0 + \beta dx^1 \right)} \\
\Rightarrow \beta'' &= \frac{\beta + \beta'}{1 + \beta\beta'}
\end{aligned}
\tag{3}
$$

The velocity is not an additive quantity. *i.e.* **non-linear in successive transformation.** Here comes the need of *"Rapidity"* to circumvent this drawback, by defining

$\beta = tanh\ y$ or $y = \frac{1}{2} ln \left(\frac{1+\beta}{1-\beta} \right)$. One can show (we will be showing this in subsequent sections) that rapidity is an additive quantity *i.e.*

$$\boxed{y'' = y + y'}$$

Using the rapidity, a Lorentz transformation with finite η, can be decomposed into N successive transformations with rapidity

$\Delta y = y/N$

Solving β, γ in terms of y, we have

$\beta = tanh\ y,\ \gamma = cosh\ y,\ \beta\gamma = sinh\ y$

Lorentz boost given by Eqn. 1 can be rewritten as

$$x^{0\prime} = (cosh\ y)\ x^0\ +\ (sinh\ y)\ x^1,$$
$$x^{1\prime} = (sinh\ y)\ x^0\ +\ (cosh\ y)\ x^1, \quad (4)$$

Comparing this with rotation in the $x - y$ plane:

$$x' = xcos\ \theta\ -\ ysin\ \theta,$$
$$y' = xsin\ \theta\ +\ ycos\ \theta, \quad (5)$$

Eqn. 4 can be obtained from Eqn. 5 by substituting

- $\theta \to -iy$ (this y is rapidity variable)

- $x \to ix^0$

- $y \to x^1$ (this y is the Cartesian coordinate)

Lorentz boost (in the x-direction) is formally a rotation by an angle $(-iy)$ in the x and imaginary time (ix^0) plane.

Experimental Consideration: In high-energy collider experiments, the secondary particles which are produced from the interaction, are boosted in the z-direction (along the beam axis). The boosted angular distribution is better expressed as rapidity distribution. At high-energies, each particle has $E \sim pc$, $p_{II} = pcos\ \theta$, and its rapidity is approximated by so-called pseudo-rapidity:

$$\eta' = \frac{1}{2}\ ln\left(\frac{1 + \beta_{II}}{1 - \beta_{II}}\right)\ =\ \frac{1}{2}\ ln\left(\frac{E + p_{II}c}{E - p_{II}c}\right)\ \sim -ln\ tan\ \theta/2. \quad (6)$$

This fact is taken into account in designing detectors, which are divided into modules that span the same solid angle in the $\eta - \phi$ (azimuthal angle) plane.

C.   Four Vectors

Th position-time 4-vector: x^μ, $\mu = 0, 1, 2, 3$; with $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$.

$$I \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$
$$= (x^{0\prime})^2 - (x^{1\prime})^2 - (x^{2\prime})^2 - (x^{3\prime})^2 \quad (7)$$

I is called the 4-dimensional length element, which is Lorentz Invariant (LI). A quantity having same value in all inertial frames is called an "invariant". This is like $r^2 = x^2 + y^2 + z^2$

being invariant under spatial rotation. I could be written in the form of a sum:

$$I = \sum_{\mu=0}^{3} x^{\mu} x^{\mu} \tag{8}$$

To take care of the negative signs, let's define a "metric" $g_{\mu\nu}$ such that

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{9}$$

Now

$$I = g_{\mu\nu}\, x^{\mu} x^{\nu} \tag{10}$$

Define covariant 4-vector x_{μ} (index down):

$$x_{\mu} = g_{\mu\nu}\, x^{\nu} \tag{11}$$

x^{μ} (index up) is called "contravariant 4-vector". With the above definitions, now

$$I = x_{\mu}x^{\mu} = x^{\mu}x_{\mu} \tag{12}$$

To each contravariant 4-vector a^{μ}, a covariant 4-vector could be assigned and vice-versa.

$$a^{\mu} = g^{\mu\nu} a_{\nu} \tag{13}$$

$$a_{\mu} = g_{\mu\nu} a^{\nu} \tag{14}$$

$g^{\mu\nu}$ are the elements in g^{-1}. Since $g^{-1} = g$, $\boxed{g^{\mu\nu} = g_{\mu\nu}}$.
Given any two 4-vectors, a^{μ} and b^{μ},

$$a^{\mu}b_{\mu} = a_{\mu}b^{\mu} = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 \tag{15}$$

is *L.I.*. The above operation is called "4-vector scalar product". Remember Einstein's summation convention (repeated Greek indices are to be summed).

$$a.b \equiv a_{\mu}b^{\mu}$$
$$= a^0 b^0 - \vec{a}.\vec{b} \tag{16}$$

$$a^2 \equiv a.a = (a^0)^2 - \vec{a}^2 \tag{17}$$

The 1st term is called "*temporal*" and the 2nd is called *spatial component*.

- If $a^2 > 0$: a^{μ} is called *time-like*. Events are in the forward light-cone. They appear later than the origin, O. Events in the backward light-cone appear earlier to O. Only events in the backward light-cone can influence O. And O can have an influence only on the events in the forward cone.

- If $a^2 < 0$: a^μ is called *space-like*. Events are called space-like events and there is no interaction with O. This is related to "*causality*".

- If $a^2 = 0$: a^μ is called *light-like*. Connects all those events with the origin which can be reached by a light signal.

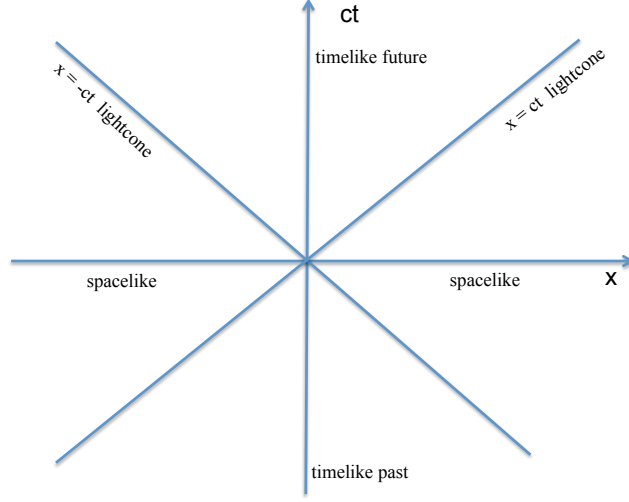This is shown pictorially in Figure 2.

FIG. 2: A schematic of lightcone diagram.

D.   Energy-Momentum Four-Vector

The velocity of a particle is given by

$$\vec{v} = \frac{d\vec{x}}{dt} \tag{18}$$

where $d\vec{x}$ is the distance travelled in the laboratory frame and dt is the time measured in the same frame. Proper velocity of the particle is given by

$$\vec{\eta} = \frac{d\vec{x}}{d\tau} \tag{19}$$

where $d\vec{x}$ is the distance travelled in the laboratory frame and $d\tau$ is the proper time. Now

$$\begin{aligned}
\vec{\eta} = \frac{d\vec{x}}{d\tau} &= \frac{d\vec{x}}{dt}\frac{dt}{d\tau} \\
&= \vec{v}\gamma \\
&\Rightarrow \boxed{\vec{\eta} = \gamma\vec{v}}
\end{aligned} \tag{20}$$

It is easy to work with the proper velocity, $\vec{\eta}$, as only $d\vec{x}$ transforms under Lorentz transformation. Furthermore,

$$\eta^\mu = \frac{dx^\mu}{d\tau} \tag{21}$$

so

$$\eta^0 \;=\; \frac{dx^0}{d\tau} \;=\; \frac{d(ct)}{\frac{1}{\gamma}dt} \;=\; \gamma c \tag{22}$$

Hence

$$\boxed{\eta^\mu \;=\; \gamma\,(c,\; v_x,\; v_y,\; v_z)} \tag{23}$$

This is called the *proper velocity 4-vector*. Remember that the spatial component brings up the negative sign for covariant tensor. Now

$$\begin{aligned}
\eta^\mu \eta_\mu &= \gamma^2\left(c^2 - v_x^2 - v_y^2 - v_z^2\right)\\
&= \gamma^2 c^2 \left(1 - \frac{v^2}{c^2}\right)\\
&= \gamma^2 c^2 \left(1 - \beta^2\right)\\
&= c^2,
\end{aligned} \tag{24}$$

which is Lorentz Invariant. This also proves that 4-vector scalar product is L.I.

We know *momentum = mass × velocity*. And velocity can be "ordinary velocity" or "proper velocity". Classically, both are equal (non-relativistic limit). If $\vec{p} \;=\; m\vec{v}$, the conservation of momentum is inconsistent with the principle of relativity. In relativity, momentum is the product of mass and proper velocity.

$$\vec{p} \equiv m\vec{\eta} \tag{25}$$

$$p^\mu \;=\; m\eta^\mu \tag{26}$$

The spatial component of p^μ constitutes the (relativistic) momentum 4-vector:

$$\vec{p} \;=\; \gamma m\vec{v} \;=\; \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}} \tag{27}$$

$$\boxed{p^0 \;=\; \gamma mc} \tag{28}$$

Relativistic energy, E:

$$E \;\equiv\; \gamma mc^2 \;=\; \frac{mc^2}{\sqrt{1 - v^2/c^2}} \tag{29}$$

Hence,

$$p^0 \;=\; \frac{E}{c} \tag{30}$$

and the energy-momentum 4-vector:

$$\boxed{p^\mu \;=\; \left(\frac{E}{c},\; p_x,\; p_y,\; p_z\right)} \tag{31}$$

Now

$$
\begin{aligned}
p^\mu p_\mu &= \frac{E^2}{c^2} - \vec{p}^{\,2} \;=\; m^2 c^2 : L.I. \\
&= (m\eta^\mu)\,(m\eta_\mu) \\
&= m^2\,(\eta^\mu \eta_\mu) \\
&= m^2 c^2 \\
&\Rightarrow \boxed{p^\mu p_\mu = m^2}\;(Natural\;Units).
\end{aligned}
\tag{32}
$$

$$
\Rightarrow E^2 \;=\; \vec{p}^{\,2} c^2 + m^2 c^4
$$

In natural units,

$$
\boxed{E^2 \;=\; \vec{p}^{\,2} + m^2}
\tag{33}
$$

- $p^2 \;=\; m^2 \;>\; 0$: Ordinary massive particle

- $p^2 \;=\; m^2 \;=\; 0$: Massless particles like photons, gravitons etc.

- $p^2 \;<\; 0$: Tachyon or virtual particles

- $p^\mu \;=\; 0$: Vacuum

Remember that the relativistic equations $\vec{p} \;=\; \gamma m \vec{v}$ and $E \;=\; \gamma m$ do not hold good for massless particles and $m \;=\; 0$ is allowed only if the particle travels with the speed of light. For massless particles,

$v \;=\; c$ and $E \;=\; |\vec{p}|c$.

E. The Choice of Units

We know

$$
x^2 \;=\; c^2 t^2 - x_1^2 - x_2^2 - x_3^2
\tag{34}
$$
$$
p^2 \;=\; m_0^2 c^2
\tag{35}
$$

The velocity of light ″c″ appears directly in these and many other formulas. Furthermore, de Broglie relation between 4-momentum and wave vector of a particle is

$$
E \;=\; \hbar\omega \;\; (obtained\;from\;Einstein's\;equation)
\tag{36}
$$

In 4-vector notation,

$$
P \;=\; \hbar K
\tag{37}
$$

where $P \;=\; \left\{\frac{E}{c},\, p\right\}$, $K \;=\; \left\{\frac{\omega}{c},\, k\right\}$. If we choose a system of unit in which

$$
\boxed{\hbar \;=\; c = 1},
$$

where $\hbar \;=\; \frac{h}{2\pi} \;=\; 1.055 \times 10^{-34}\;Joule\;sec$: unit of action/angular momentum (ML^2/T).

$c = 2.998 \times 10^8 \ meter \ sec^{-1}$: unit of velocity, the velocity of light in vacuum (L/T).

Now the relativistic formula for energy,

$$E^2 = p^2 c^2 + m_0^2 c^4 \tag{38}$$

in this new system of unit (called natural unit and more popularly used in high-energy (particle) physics) becomes

$$\boxed{E^2 = p^2 + m_0^2} \tag{39}$$

We can define system of units completely, if we specify the unit of energy (ML^2/T^2). In particle physics, unit of energy is GeV ($1 \ GeV = 10^9 \ eV$). This choice is motivated by the rest mass of proton $\sim 1 \ GeV$. This gives rise to mass (m), momentum (mc), energy (mc^2) in GeV. Length ($\frac{\hbar}{mc}$) and time ($\frac{\hbar}{mc^2}$) in GeV^{-1}.

Taking the values of $\hbar = c = 1$, one obtains,

$$\boxed{1 \ sec = 1.52 \times 10^{24} \ GeV^{-1}}$$

$$\boxed{1 \ meter = 5.07 \times 10^{15} \ GeV^{-1}}$$

$1 \ fermi \equiv 1 \ fm = 10^{-13} \ cm = 10^{-15} m$

$$\Rightarrow \boxed{1 \ fm = 5.07 \ GeV^{-1}}$$

$$\boxed{1 \ fm = 3.33 \times 10^{-24} \ sec}$$

$$\boxed{197 \ MeV = 1 \ fm^{-1}}$$

Note: $1 \ TeV = 10^3 \ GeV = 10^6 \ MeV = 10^9 \ KeV = 10^{12} \ eV$.

The additional advantage of using natural unit in high energy particle physics is that we deal with strong interaction, whose lifetime $\sim 10^{-24} \ sec$, the decay length of particle can be better expressed in terms of fermi.

F. Collider Vs Fixed Target Experiment

1. For Symmetric Collisions $(A + A)$

Consider the collision of two particles. In LS, the projectile with momentum $\mathbf{p}_1$, energy E_1 and mass m_1 collides with a particle of mass m_2 at rest. The 4-momenta of the particles are

$p_1 = (E_1, \mathbf{p}_1), \qquad p_2 = (m_2, \mathbf{0})$

In CMS, the momenta of both the particles are equal and opposite, the 4-momenta are

$$p_1^* = (E_1^*, \mathbf{p}_1^*), \qquad p_2^* = (E_2^*, -\mathbf{p}_1^*)$$

The total 4-momentum of the system is a conserved quantity in the collision.

In CMS,

$$
\begin{aligned}
p_\mu p^\mu = (p_1 + p_2)^2 &= (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 \\
&= (E_1 + E_2)^2 \\
&= E_{cm}^2 \equiv s
\end{aligned}
\tag{40}
$$

$\sqrt{s}$ is the total energy in the CMS which is the invariant mass of the CMS.

In LS,
$$p_\mu p^\mu = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2E_1 m_2 \tag{41}$$

Hence

$$\boxed{E_{cm} = \sqrt{s} = \sqrt{m_1^2 + m_2^2 + 2E_{proj} m_2}} \tag{42}$$

where $E_1 = E_{proj}$, the projectile energy in LS. Hence it is evident here that the CM frame with an invariant mass $\sqrt{s}$ moves in the laboratory in the direction of $\mathbf{p}_1$ with a velocity corresponding to:
Lorentz factor,

$$\gamma_{cm} = \frac{E_1 + m_2}{\sqrt{s}} \tag{43}$$

$$\Rightarrow \sqrt{s} = \frac{E_{lab}}{\gamma_{cm}}, \tag{44}$$

this is because $E = \gamma m$ and

$$y_{cm} = cosh^{-1} \gamma_{cm}. \tag{45}$$

The center of mass or center of momentum frame (CM/CMS) is at rest and the total momentum is zero. This makes it a suitable choice for solving kinematics problems.

Note: We know that for a collider with head-on collision ($\theta = 180^0$)

$$s = E_{cm}^2 = m_1^2 + m_2^2 + 2(E_1.E_2 + |\mathbf{p}_1||\mathbf{p}_2|) \tag{46}$$

For relativistic collisions, $m_1, \ m_2 \ll E_1, E_2$

$$E_{cm}^2 \simeq 4E_1 E_2 \tag{47}$$

For two beams crossing at an angle θ,

$$E_{cm}^2 = 2E_1 E_2 (1 + cos \ \theta) \tag{48}$$

The CM energy available in a collider with equal energies (E) for new particle production rises linearly with E *i.e.*

$$E_{cm} \simeq 2E \tag{49}$$

For a fixed-target experiment the CM energy rises as the square root of the incident energy:

$$E_{cm} \simeq \sqrt{2m_2 E_1} \tag{50}$$

Hence the highest energy available for new particle production is achieved at collider experiments. For example, at SPS fixed-target experiment to achieve a CM energy of 17.3 AGeV the required incident beam energy is 158 AGeV.

Problem: Suppose two identical particles, each with mass m and kinetic energy T, collide head-on. What is their relative kinetic energy, T' (*i.e.* K.E. of one in the rest frame of the other). Apply this to an electron-positron collider, where K.E. of electron (positron) is 1 GeV. Find the K.E. of electron if positron is at rest (fixed target). Which experiment is preferred, a collider or a fixed-target experiment?

<u>Note:</u> Most of the times the energy of the collision is expressed in terms of nucleon-nucleon center of mass energy. In the nucleon-nucleon CM frame, two nuclei approach each other with the same boost factor γ. The nucleon-nucleon CM is denoted by $\sqrt{s_{NN}}$ and is related to the total CM energy by

$$\sqrt{s} = A \sqrt{s_{NN}} \tag{51}$$

This is for a symmetric collision with number of nucleons in each nuclei as A. The colliding nucleons approach each other with energy $\sqrt{s_{NN}}/2$ and with equal and opposite momenta. The rapidity of the nucleon-nucleon center of mass is
$y_{NN} = 0$ and taking $m_1 = m_2 = m_p$, the projectile and target nucleons are at equal and opposite rapidities.

$$y_{proj} = -y_{target} = cosh^{-1} \frac{\sqrt{s_{NN}}}{2m_p} = y_{beam}. \tag{52}$$

<u>Note:</u> Lorentz Factor

$$\begin{aligned}
\gamma &= \frac{E}{M} = \frac{\sqrt{s}}{2A \, m_p} \\
&= \frac{A \sqrt{s_{NN}}}{2A \, m_p} = \frac{\sqrt{s_{NN}}}{2 \, m_p} \\
&= \frac{E_{beam}^{CMS}}{m_p}
\end{aligned} \tag{53}$$

where E and M are Energy and Mass in CMS respectively. Assuming mass of a proton, $m_p \sim 1$ GeV, the Lorentz factor is of the order of beam energy in CMS for a symmetric collision.

2. *For Asymmetric Collisions (A + B)*

During the early phase of relativistic nuclear collision research, the projectile mass was limited by accelerator-technical conditions (^{38}Ar at the Bevalac, ^{28}Si at the AGS, ^{32}S at the
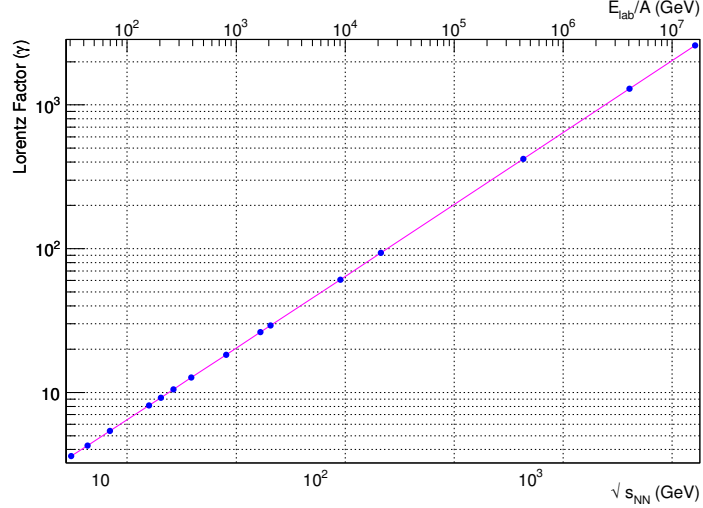
FIG. 3: A plot to demonstrate how the Lorentz Factor increases with collision energy for a symmetric collision. Center of mass system is compared with fixed target experiment.

SPS). Nevertheless, collisions with mass ≈ 200 nuclear targets were investigated. Analysis of such collisions is faced with the problem of determining an "effective" center of mass frame, to be evaluated from the numbers of projectile and target participant nucleons, respectively. Their ratio - an thus the effective CM rapidity - depends on impact parameter. Moreover, this effective CM frame refers to soft hadron production only, whereas hard processes are still referred to the frame of nucleon-nucleon collisions. The light projectile on heavy target kinematics are described in [17].

The center of mass energy of a collision of two different systems with charge Z_1, Z_2 and atomic numbers A_1, A_2 with $Z = A = 1$, for a proton is

$$\boxed{\sqrt{s_{NN}} \simeq 2\sqrt{s_{pp}} + \sqrt{s_{pp}}\sqrt{\frac{Z_1 Z_2}{A_1 A_2}}} \tag{54}$$

where sub-index NN refers to the energy per nucleon inside the colliding nucleus and $\sqrt{s_{pp}}$ is the corresponding energy in pp collisions. The rapidity shift in non-symmetric systems is given by

$$\boxed{\Delta y \simeq \frac{1}{2} \, ln \left[\frac{Z_1 A_2}{Z_2 A_1} \right]} \tag{55}$$

This is due to the fact that the center-of-mass frame of the pA collision doesn't coincide with the laboratory center-of-mass frame. The rapidity shift in $p + Pb$ collision is

$$\begin{aligned}
\Delta y &\simeq \frac{1}{2} \, ln \left[\frac{Z_1 A_2}{Z_2 A_1} \right] \\
&= \frac{1}{2} \, ln \left[\frac{82 \times 1}{1 \times 208} \right] \\
&= -0.465
\end{aligned}$$

Hence $\Delta y = \pm 0.465$ for $p + Pb$ collisions, flipping the beams. This rapidity shift need to be taken into account for the comparison with Pb+Pb data.
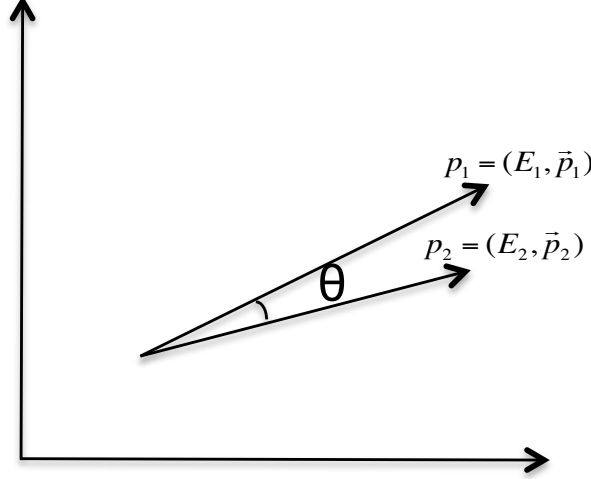
FIG. 4: The Laboratory system: two particle kinematics.

At LHC, the maximum proton beam energy is 7 TeV, while the maximum Pb beam energy is 2.75 TeV, $\sqrt{s} = 8.775$ TeV. The difference in available energy is due to the charge-to-mass ratio, Z/A. More is the number of neutrons in the nucleus, difficult it is to accelerate to higher energies. Because of different energies, the two beams will also not have the same rapidity. For the proton beam $y_p = 9.61$ and for the Pb beam it is $y_{Pb} = 8.67$. Thus the center of the collision is shifted away from $y_{cm} = 0$ by $\Delta y_{cm} = (y_p - y_{Pb})/2 = 0.47$.

G.   The Energy and Velocity of the Center of Momentum

As discussed in Ref. [2], lets consider a Lorentz system- call it the laboratory system (frame), with two particles with masses, m_1 and m_2 and four momenta p_1 and p_2, respectively as shown in Fig. 4.

What is the centre of mass energy, E ?

1. It is independent of the Lorentz system, where p_1 and p_2 are defined.

2. It must be possible to have an answer in terms of the three invariants, namely
   $p_1^2 = m_1^2$ and $p_2^2 = m_2^2$
   and $[p_1 p_2$ or $(p_1 + p_2)^2$ or $(p_1 - p_2)^2]$.

The answer would be trivial in the center of momentum (mass) frame itself. We use asterisk (*) in the CM frame to distinguish that from the laboratory frame.

In CM frame,

$$(\vec{p_1}^* + \vec{p_2}^*) = 0$$
$$\implies p_1 + p_2 = (E_1^* + E_2^*, \vec{0})$$
$$and \ E^* = E_1^* + E_2^* \tag{56}$$

Hence,

$$
\begin{aligned}
(E^*)^2 &= (E_1^* + E_2^*)^2 \\
&= (p_1^* + p_2^*)^2 \\
&= (p_1 + p_2)^2.
\end{aligned}
\tag{57}
$$

This is because $(p_1 + p_2)^2$ is invariant.

Let us define the total mass M of the system as the square of the total 4-momentum as:

$$
\boxed{M^2 = (p_1 + p_2)^2 = P^2 = (E^*)^2 = (E_1 + E_2)^2 - (p_1 + p_2)^2 = Invariant}
$$

Kinematically, for the given system, the two particles p_1 and p_2 are equivalent to one single particle with 4-momentum P and mass $M = E_{\mathrm{CM}}$. Generalizing this, one can consider these individual particles to represent a system of particles.

Further, we know

$$
\boxed{\vec{p} = m\vec{v}\gamma, \text{ and } E = m\gamma}, \text{ where } \gamma = \frac{1}{\sqrt{1-\beta^2}}.
$$

Hence, the 4-momentum of the two-particle system is given by:

$$
\vec{P} = M\vec{\beta}\gamma
$$

$$
E = M\gamma.
$$
Using the above, one obtains:

$$
\begin{aligned}
\beta_{\mathrm{CM}} &= \frac{\vec{P}}{E} \tag{58} \\
&= \frac{(\vec{p_1} + \vec{p_2})}{(E_1 + E_2)}, \tag{59}
\end{aligned}
$$

is the velocity of the CM seen from the laboratory system.

$$
\begin{aligned}
\gamma_{\mathrm{CM}} &= \frac{1}{\sqrt{1 - \beta^2}} \\
&= \frac{E}{M} \tag{60} \\
&= \frac{E_1 + E_2}{\sqrt{(E_1 + E_2)^2 - (\vec{p_1} + \vec{p_2})^2}} \\
&= \frac{E_1 + E_2}{E_{\mathrm{CM}}}, \tag{61}
\end{aligned}
$$

is the Lorentz factor or the Lorentz boost of the CM. In general, *the Lorentz factor of the CM is the ratio of the sum of the energies of the particles in the laboratory system and the energy of the CM.*

H. The Energy, Momentum and Velocity of one particle as seen from the rest system of another one

Lets assume that we sit on particle 1, which is in motion. What will be the energy of particle 2, for us?

The answer to this question must always be the same, irrespective of the Lorentz system we start with. It is thus expressible by the invariants discussed in the preceding subsection (point-2), where the last two variables are the well-known Mandelstam's s and t variables.

Let E_{21}: is the energy of particle 2, if we look at it sitting on particle 1, which then appears to be at rest for us.
$E_{21} = E_2$, in the system where $\vec{p_1} = 0$. One needs to write E_{21} in terms of the invariants available in this problem.

Now, $p_1 p_2 = E_1 E_2 - \vec{p_1}.\vec{p_2} = m_1 E_2$ (since $\vec{p_1} = 0$).

Hence, $E_{21} = E_2 = \frac{p_1 p_2}{m_1}$.

Note that the RHS is already in an invariant form.

$|\vec{p}_{21}|^2 = E_{21}^2 - m_2^2 = \frac{(p_1 p_2)^2 - m_1^2 m_2^2}{m_1^2}$.

If p_1 and p_2 are the momentum 4-vectors of any two particles in any Lorentz system, then

$$E_{21} = \frac{p_1 p_2}{m_1}$$

$$|\vec{p}_{21}|^2 = \frac{(p_1 p_2)^2 - m_1^2 m_2^2}{m_1^2} \qquad (p_1 p_2 \equiv E_1 E_2 - \vec{p_1}\vec{p_2})$$

$$v_{21}^2 = \frac{|\vec{p}_{21}|^2}{E_{21}^2} = \frac{(p_1 p_2)^2 - m_1^2 m_2^2}{(p_1 p_2)^2} \qquad (62)$$

The above three equations give the energy, momentum and velocity of particle 2, as seen from particle 1. The velocity v_{21} is the relative velocity, which is symmetric in 1 and 2. Note here that all these expressions are invariant and can be evaluated in any Lorentz system.

I. The Energy, Momentum, and Velocity of a Particle as seen from the CM System

This problem is now like all the above quantities are as seen from a *fictitious particle M*, called the "*center-of-momentum-particle*", whose 4-momentum is

$$P = p_1 + p_2 \qquad (63)$$

We need to apply formulae (Eq. 62) with p_1 replaced by P and p_2 by the 4-momentum

of that particle whose energy, momentum and velocity, we like to determine. From Eq. 62,

$$E_1^* = \frac{Pp_1}{M}$$

$$|\vec{p}_1^*|^2 = \frac{(Pp_1)^2 - M^2 m_1^2}{M^2}$$

$$v_1^{*2} = \frac{(Pp_1)^2 - M^2 m_1^2}{(Pp_1)^2} \tag{64}$$

This is by using Eq. 63 and by using

$$
\begin{aligned}
p_1 p_2 &= \frac{1}{2}[(p_1 + p_2)^2 - p_1^2 - p_2^2] \\
&= \frac{1}{2}(M^2 - m_1^2 - m_2^2),
\end{aligned} \tag{65}
$$

one obtains

$$E_1^* = \frac{M^2 + (m_1^2 - m_2^2)}{2M} \tag{66}$$

$$E_2^* = \frac{M^2 - (m_1^2 - m_2^2)}{2M} \tag{67}$$

$$E_1^* + E_2^* = M \tag{68}$$

$$
\begin{aligned}
|\vec{p}^*|^2 &= |\vec{p}_1^*|^2 = |\vec{p}_2^*|^2 \\
&= \frac{M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)}{4M^2} \\
&= \frac{[M^2 - (m_1 + m_2)^2]\,[M^2 - (m_1 - m_2)^2]}{4M^2}
\end{aligned} \tag{69}
$$

$$v_1^{*2} = \left(\frac{|\vec{p}^*|}{E_1^*}\right)^2. \tag{70}$$

Here, E_1^*, v_1^* are energy and velocity of particle 1, as seen from their common CM system and $M^2 = P^2 = (p_1 + p_2)^2$ is the square of the total mass. The above equations give the energy, momenta and the velocities of two particles m_1 and m_2 for which

$$M \to m_1 + m_2$$

J. Description of Nucleus-Nucleus Collisions in terms of Light-Cone Variables

In relativistic nucleus-nucleus collisions, it is convenient to use kinematic variables which take simple forms under Lorentz transformations for the change of frame of reference. A few of them are the light cone variables x_+ and x_-, the rapidity and pseudorapidity variables, y and η. A particle is characterized by its 4-momentum, $p_\mu = (E, \mathbf{p})$. In fixed target and collider experiments where the beam(s) define reference frames, boosted along their
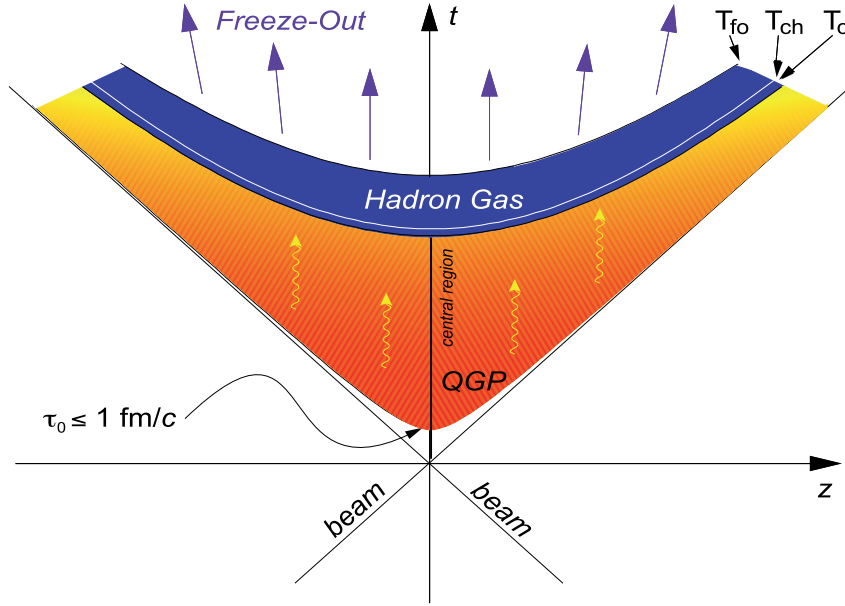
FIG. 5: Description of heavy-ion collisions in one space (z) and one time (t) dimension.

direction, it is important to express the 4-momentum in terms of more practical kinematic variables.

Figure 5 shows the collision of two Lorentz contracted nuclei approaching each other with velocities nearly equal to velocity of light. The vertical axis represents the time direction with the lower half representing time before the collision and the upper half, time after the collision. The horizontal axis represents the spatial direction. Both the nuclei collide at $(t, z) = (0, 0)$ and then the created fireball expands in time going through various processes till the created particles freeze-out and reach the detectors. The lines where $t^2 - z^2 = 0$ (note that $\sqrt{t^2 - z^2} \equiv \tau$, τ being the proper time of the particle) along the path of the colliding nuclei define the light cone. The upper part of the light-cone, where $t^2 - z^2 > 0$, is the time-like region. In nucleus-nucleus collisions, particle production occurs in the upper half of the (t, z)-plane within the light-cone. The region outside the light cone for which $t^2 - z^2 < 0$ is called space-like region. The space-time rapidity is defined as

$$\eta_s = \frac{1}{2} \, ln \left(\frac{t + z}{t - z} \right) \tag{71}$$

It could be seen that η_s is not defined in the space-like region. It takes the value of positive and negative infinity along the beam directions for which $t = \pm z$ respectively. A particle is "light-like" along the beam direction. Inside the light-cone which is time-like, η_s is properly defined.

For a particle with 4-momentum p $(p_0, \mathbf{p_T}, p_z)$, the light-cone momenta are defined by

$$p_+ = p_0 + p_z \tag{72}$$
$$p_- = p_0 - p_z \tag{73}$$

p_+ is called "*forward light-cone momentum*" and p_- is called "*backward light-cone momentum*".

For a particle travelling along the beam direction, has higher value of forward light-cone

momentum and travelling opposite to the beam direction has lower value of forward light-cone momentum. The advantages of using light-cone variables to study particle production are the following.

1. The forward light-cone momentum of any particle in one frame is related to the forward light-cone momentum of the same particle in another boosted Lorentz frame by a constant factor.

2. Hence, if a daughter particle c is fragmenting from a parent particle b, then the ratio of the forward light-cone momentum of c relative to that of b is independent of the Lorentz frame.

Define

$$
\begin{aligned}
x_+ &= \frac{p_0^c + p_z^c}{p_0^b + p_z^b} \\
&= \frac{c_+}{b_+}.
\end{aligned}
\tag{74}
$$

The forward light-cone variable x_+ is always positive because c_+ can't be greater than b_+. Hence the upper limit of x_+ is 1. x_+ is Lorentz invariant.

3. The Lorentz invariance of x_+ provides a tool to measure the momentum of any particle in the scale of the momentum of any reference particle.

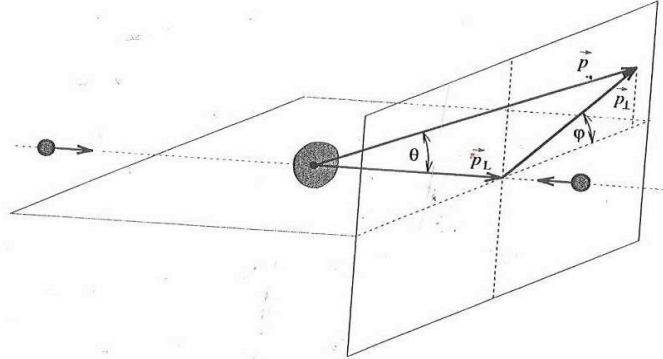### K. Pictorial Representation of Detector System

FIG. 6: A schematic decomposition of particle momentum $\vec{p}$ (in CM frame) into parallel and longitudinal components. Note the angle of inclination θ of $\vec{p}$ and the azimuthal angle ϕ of $p_\perp$ [3].

In an collider experimental environment, a particle is emitted from the collision point making a polar angle θ with the collision point. When the momenta of the particles are determined by a tracking detector, the (pseudo)-rapidity and azimuthal angles are given by

$$y = tanh^{-1} v_z = tanh^{-1}\frac{p_z}{E} = tanh^{-1}\frac{p_z}{\sqrt{p_x^2 + p_y^2 + p_z^2 + m_0^2}}$$

and $\boxed{\phi = tan^{-1}\frac{p_y}{p_x}}$.

The polar angle θ is given by

$$\theta = \cos^{-1}\frac{p_z}{|\vec{p}|} = \tan^{-1}\frac{|\vec{p_T}|}{p_z}.$$

Pictorially, these are shown in the Figure 6. A detector plane is spanned by (η, ϕ), with η decreasing while going away from the beam axis in annular rings and ϕ is scanned making an angle with the beam axis and increasing it anti-clock-wise, as is shown in the picture.

L. The Rapidity Variable

One of the important tasks in hand is to introduce key kinematic variables that relate particle momentum to the dynamics that is occurring in heavy-ion collisions. It is essentially the convenience of working in center-of-momentum system, we need to introduce the observable rapidity. To do so, let's proceed as follows: The co-ordinates along the beam line (conventionally along the z-axis) is called *longitudinal* and perpendicular to it is called *transverse* (x-y). The 3-momentum can be decomposed into the longitudinal (p_z) and the transverse ($\mathbf{p}_T$), $\mathbf{p}_T$ being a vector quantity which is invariant under a Lorentz boost along the longitudinal direction. The variable rapidity "y" is defined by

$$y = \frac{1}{2}ln\left(\frac{E + p_z}{E - p_z}\right) \tag{75}$$

$$= ln\left(\frac{E + p_z}{m_T}\right) \tag{76}$$

It is a dimensionless quantity related to the ratio of forward light-cone to backward light-cone momentum. The rapidity changes by an additive constant under longitudinal Lorentz boosts.

For a free particle which is on the mass shell (for which $E^2 = p^2 + m^2$), the 4-momentum has only three degrees of freedom and can be represented by $(y, \mathbf{p}_T)$. $(E, \mathbf{p}_T)$ could be expressed in terms of $(y, \mathbf{p}_T)$ as

$$E = m_T \ cosh \ y \tag{77}$$
$$p_z = m_T \ sinh \ y \tag{78}$$

m_T being the transverse mass which is defined as

$$m_T^2 = m^2 + \mathbf{p}_T^2. \tag{79}$$

The advantage of rapidity variable is that the shape of the rapidity distribution remains unchanged under a longitudinal Lorentz boost. When we go from CMS to LS, the rapidity distribution is the same, with the y-scale shifted by an amount equal to y_{cm}. This is shown below.

1. Rapidity of Center of Mass in the Laboratory System

The total energy in the CMS system is $E_{cm} = \sqrt{s}$. The energy and momentum of the CMS in the LS are $\gamma_{cm}\sqrt{s}$ and $\beta_{cm}\gamma_{cm}\sqrt{s}$ respectively. The rapidity of the CMS in the LS

is

$$y_{cm} = \frac{1}{2} \, ln \left[\frac{\gamma_{cm}\sqrt{s} + \beta_{cm}\gamma_{cm}\sqrt{s}}{\gamma_{cm}\sqrt{s} - \beta_{cm}\gamma_{cm}\sqrt{s}} \right]$$
$$= \frac{1}{2} \, ln \left[\frac{1 + \beta_{cm}}{1 - \beta_{cm}} \right] \tag{80}$$

It is a constant for a particular Lorentz transformation.

2. Relationship between Rapidity of a particle in LS and rapidity in CMS

The rapidities of a particle in the LS and CMS of the collision are respectively, $y = \frac{1}{2} \, ln \left(\frac{E+p_z}{E-p_z} \right)$ and $y^* = \frac{1}{2} \, ln \left(\frac{E^*+p_z^*}{E^*-p_z^*} \right)$. For a particle travelling in longitudinal direction, the Lorentz transformation of its energy and momentum components give

$$\begin{bmatrix} E^* \\ P_L^* \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \cdot \begin{bmatrix} E \\ P_L \end{bmatrix}, P_T^* = P_T \tag{81}$$

where P_L and P_T are the longitudinal and transverse components of $\vec{P}$, which are parallel and perpendicular to β, respectively. Hence, the inverse Lorentz transformations on E and p_z give

$$y = \frac{1}{2} \, ln \left[\frac{\gamma(E^* + \beta p_z^*) + \gamma(\beta E^* + p_z^*)}{\gamma(E^* + \beta p_z^*) - \gamma(\beta E^* + p_z^*)} \right]$$
$$= \frac{1}{2} \, ln \left[\frac{E^* + p_z^*}{E^* - p_z^*} \right] + \frac{1}{2} \, ln \left[\frac{1 + \beta}{1 - \beta} \right] \tag{82}$$
$$\Rightarrow y = y^* + y_{cm}. \tag{83}$$

Hence the rapidity of a particle in the laboratory system is equal to the sum of the rapidity of the particle in the center of mass system and the rapidity of the center of mass in the laboratory system. It can also be state that the rapidity of a particle in a moving (boosted) frame is equal to the rapidity in its own rest frame minus the rapidity of the moving frame. In the non-relativistic limit, this is like the subtraction of velocity of the moving frame. However, this is not surprising because, non-relativistically, the rapidity y is equal to longitudinal velocity β. Rapidity is a relativistic measure of the velocity. This simple property of the rapidity variable under Lorentz transformation makes it a suitable choice to describe the dynamics of relativistic particles. The simple shape invariance nature of rapidity spectra brings its importance in the analysis of particle production in nuclear collisions. For instance, in fixed-target experiments, we can study particle spectra using y as a variable without making an explicit transformation to the CM frame of reference and from the rapidity spectra, we deduce the point of symmetry corresponding to the CM rapidity. In symmetric collisions with fixed targets, the CM frame is located in the middle between the rapidities of the projectile and target *i.e.* $y_{cm} = y_{proj}/2$. In this case, the particle rapidity spectrum must be symmetric around y_{cm}. This allows for complementing measured particle spectra: if these are available for, *e.g.* $y \geq y_{cm}$, a reflection at the symmetry point y_{cm} gives us the part of the spectrum with $y \leq y_{cm}$, for which an experimental measurement is absent.
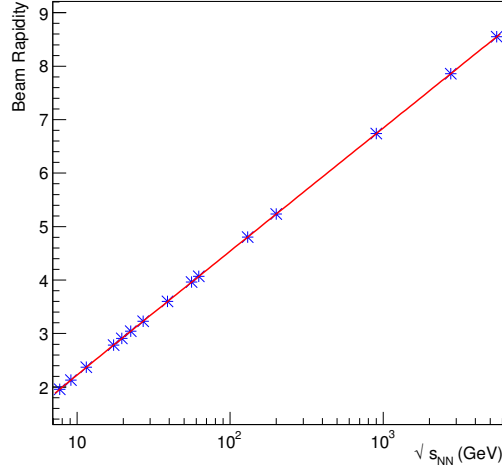
FIG. 7: A plot of beam rapidity as a function of center of mass energy.

3. *Relationship between Rapidity and Velocity*

Consider a particle travelling in z-direction with a longitudinal velocity β. The energy E and the longitudinal momentum p_z of the particle are

$$E = \gamma m \tag{84}$$
$$p_z = \gamma\beta m \tag{85}$$
$$p_z = \beta E \tag{86}$$

where m is the rest mass of the particle. Hence the rapidity of the particle travelling in z-direction with velocity β is

$$
\begin{aligned}
y_\beta &= \frac{1}{2}\, ln\left[\frac{E+p_z}{E-p_z}\right] = \frac{1}{2}\, ln\left[\frac{\gamma m + \gamma\beta m}{\gamma m - \gamma\beta m}\right] \\
&= \frac{1}{2}\, ln\left[\frac{1+\beta}{1-\beta}\right]
\end{aligned}
\tag{87}
$$

Note here that y_β is independent of particle mass. A particle is said to be relativistic in nature if $\gamma >> 1$, $\beta \approx 1$ and $E \approx p$. In other words, the energy of the particle is much higher than its rest mass i.e. $E >> m_0$ and hence the energy and total momentum of the particle are comparable. In the non-relativistic limit when β is small, expanding y_β in terms of β leads to

$$y_\beta = \beta + O(\beta^3) \tag{88}$$

Thus the rapidity of the particle is the relativistic realization of its velocity.

4. *Beam Rapidity*

We know,
$E = m_T \cosh y$, $p_z = m_T \sinh y$ and $m_T^2 = m^2 + \mathbf{p}_T^2$.

For the beam particles, $p_T = 0$.

Hence, $E = m_b \cosh y_b$ and $p_z = m_b \sinh y_b$,

where m_b and y_b are the rest mass and rapidity of the beam particles.

$$\boxed{y_b = \cosh^{-1}(E/m_b)} \tag{89}$$

Note that Eq. 89 is used for fixed target experiment with E as the energy of individual beam particles, neglecting the rest mass. For example, for SPS beam energy of 158 AGeV, $E = 158$ GeV. This leads to $y_b = 5.82$ in fixed target framework. Now when we convert Eq. 89 for a collider system, we get

$$\begin{aligned} y_b &= \cosh^{-1}(E/m_b) \\ &= \cosh^{-1}\left[\frac{\sqrt{s_{NN}}}{2\,m_p}\right] \end{aligned} \tag{90}$$

$$\Rightarrow \boxed{y_b = \ln(\sqrt{s_{NN}}/m_p)} \tag{91}$$

Now, for the above SPS beam energy of 158 AGeV, the equivalent center of mass energy per nucleon is, $\sqrt{s_{\mathrm{NN}}} = 17.3$ GeV. By using Eq. 91, we get $y_b = 2.91$. This makes sense to us, as in a collider experiment both the beams come in opposite directions with equal beam rapidities.

Further,

$$y_b = \sinh^{-1}(p_z/m_b) \tag{92}$$

In terms of the beam velocity, beam rapidity can be expressed as

$$y_b = \tanh^{-1}(p_z/E) = \tanh^{-1}\beta \tag{93}$$

Here m_p is the mass of a proton. Note that the beam energy $E = \sqrt{s_{NN}}/2$ for a symmetric collider.

Furthermore, it could be shown that: $\boxed{y_b = \mp \ln(\sqrt{s_{NN}}/m_p) = \mp y_{\max}}$

Example For the nucleon-nucleon center of mass energy $\sqrt{s_{NN}} = 9.1$ GeV, the beam rapidity $y_b = \cosh^{-1}\left(\frac{9.1}{2\times 0.938}\right) = 2.26$

For p+p collisions with lab momentum 100 GeV/c,

$y_b = \sinh^{-1}\left(\frac{p_z}{m_b}\right) = \sinh^{-1}\left(\frac{100}{0.938}\right) = 5.36$

and for Pb+Pb collisions at SPS with lab energy 158 AGeV, $y_b = 2.91$.

TABLE I: Table of collision energy, $\sqrt{s_{NN}}$ Vs the beam rapidity, y_b.

$\sqrt{s_{NN}}$ (GeV)	7.7	9.1	11.5	17.3	19.6	27.0	39.0	62.4	130	200	900	2360	2760	5520	7000	14000
y_b		2.10	2.27	2.50	2.91	3.03	3.35	3.72	4.19	4.93	5.36	6.86	7.83	7.98	8.68	8.91	9.61

5.   *Rapidity of the CMS in terms of Projectile and Target Rapidities*

Let us consider the beam particle "b" and the target particle "a".
$b_z = m_T \, sinh \, y_b = m_b \, sinh \, y_b$. This is because p_T of beam particles is zero. Hence

$$y_b = sinh^{-1} \, (b_z/m_b). \tag{94}$$

The energy of the beam particle in the laboratory frame is
$b_0 = m_T \, cosh \, y_b = m_b \, cosh \, y_b$.
Assuming target particle a has longitudinal momentum a_z, its rapidity in the laboratory frame is given by

$$y_a = sinh^{-1} \, (a_z/m_a) \tag{95}$$

and its energy

$$a_0 = m_a \, cosh \, y_a. \tag{96}$$

The CMS is obtained by boosting the LS by a velocity of the center-of-mass frame β_{cm} such that the longitudinal momentum of the beam particle b_z^* and of the target particle a_z^* are equal and opposite. Hence β_{cm} satisfies the condition,
$a_z^* = \gamma_{cm}(a_z - \beta_{cm}a_0) = -b_z^* = -\gamma_{cm}(b_z - \beta_{cm}b_0)$, where $\gamma_{cm} = \frac{1}{\sqrt{1-\beta_{cm}^2}}$. Hence,

$$\beta_{cm} = \frac{a_z + b_z}{a_0 + b_0}. \tag{97}$$

We know the rapidity of the center of mass is

$$y_{cm} = \frac{1}{2} \, ln \left[\frac{1 + \beta_{cm}}{1 - \beta_{cm}} \right] \tag{98}$$

Using equations 97 and 98, we get

$$y_{cm} = \frac{1}{2} \, ln \left[\frac{a_0 + a_z + b_0 + b_z}{a_0 - a_z + b_0 - b_z} \right]. \tag{99}$$

Writing energies and momenta in terms of rapidity variables in the LS,

$$
\begin{aligned}
y_{cm} &= \frac{1}{2} \, ln \left[\frac{m_a \, cosh \, y_a + m_a \, sinh \, y_a + m_b \, cosh \, y_b + m_b \, sinh \, y_b}{m_a \, cosh \, y_a - m_a \, sinh \, y_a + m_b \, cosh \, y_b - m_b \, sinh \, y_b} \right] \\
&= \frac{1}{2}(y_a + y_b) + \frac{1}{2} \, ln \left[\frac{m_a \, e^{y_a} + m_b \, e^{y_b}}{m_a \, e^{y_b} + m_b \, e^{y_a}} \right]
\end{aligned}
\tag{100}
$$

For a symmetric collision (for $m_a = m_b$),

$$y_{cm} = \frac{1}{2}(y_a + y_b) \tag{101}$$

Rapidities of a and b in the CMS are

$$y_a^* = y_a - y_{cm} = -\frac{1}{2}(y_b - y_a) \tag{102}$$

$$y_b^* = y_b - y_{cm} = \frac{1}{2}(y_b - y_a). \tag{103}$$

Given the incident energy, the rapidity of projectile particles and the rapidity of the target particles can thus be determined. The greater is the incident energy, the greater is the separation between the projectile and target rapidity.

Central Rapidity   The region of rapidity mid-way between the projectile and target rapidities is called central rapidity.

Example In p+p collisions at a laboratory momentum of 100 GeV/c, beam rapidity $y_b = 5.36$, target rapidity $y_a = 0$ and the central rapidity ≈ 2.7.

6.   Mid-rapidity in Fixed target and Collider Experiments

In fixed-target experiments (LS), $y_{target} = 0$.

$y_{lab} = y_{target} + y_{projectile} = y_{beam}$ Hence mid-rapidity in fixed-target experiment is given by,

$$y_{mid}^{LS} = y_{beam}/2. \tag{104}$$

In collider experiments (center of mass system),

$y_{projectile} = -y_{target} = y_{CMS} = y_{beam}/2.$

Hence, mid-rapidity in CMS system is given by

$$y_{mid}^{CMS} = (y_{projectile} + y_{target})/2 = 0. \tag{105}$$

This is valid for a symmetric energy collider. The rapidity difference is given by $y_{projectile} - y_{target} = 2y_{CMS}$ and this increases with energy for a collider as y increases with energy.

7.   Light-cone variables and Rapidity

Consider a particle having rapidity y and the beam rapidity is y_b. The particle has forward light-cone variable x_+ with respect to the beam particle

$$
\begin{aligned}
x_+ &= \frac{p_0^c + p_z^c}{p_0^b + p_z^b} \\
&= \frac{m_T^c}{m^b} e^{y - y_b}
\end{aligned}
\tag{106}
$$

where m_T^c is the transverse mass of c. Note that the transverse momentum of the beam particle is zero. Hence,

$$y = y_b + ln\ x_+ + ln\left(\frac{m_b}{m_T^c}\right) \tag{107}$$

Similarly, relative to the target particle a with a target rapidity y_a, the backward light-cone variable of the detected particle c is x_-. x_- is related to y by

$$x_- = \frac{m_T^c}{m^b} e^{y_a - y} \tag{108}$$

and conversely,

$$y = y_a - ln\ x_- - ln\left(\frac{m_a}{m_T^c}\right). \tag{109}$$

In general, the rapidity of a particle is related to its light-cone momenta by

$$y = \frac{1}{2}\ ln\left(\frac{p_+}{p_-}\right) \tag{110}$$

Note that in situations where there is a frequent need to work with boosts along z-direction, it's better to use $(y, \mathbf{p_T})$ for a particle rather than using it's 3-momentum, because of the simple transformation rules for y and $\mathbf{p_T}$ under Lorentz boosts.

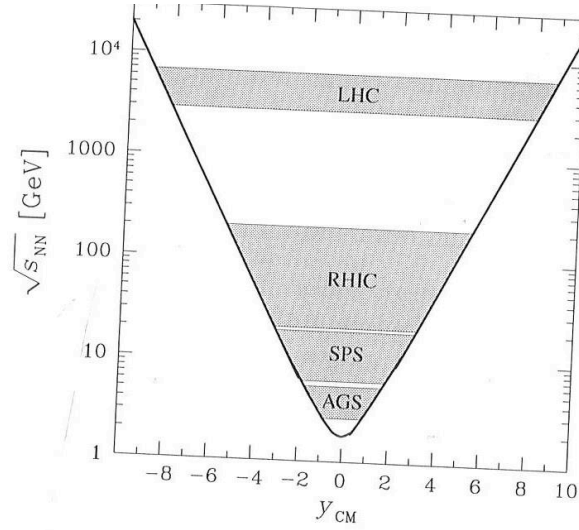8. *The Maximum Accessible Rapidity in an Interaction*

FIG. 8: $\sqrt{s_{NN}}$ (vertical axis) of various accelerators as a function of the projectile and target rapidities seen from the CM frame. Shaded areas: energy ranges accessible at various accelerators [3].

We know the variable rapidity "y" of a particle is defined by Eqn. 76:

$$y = \frac{1}{2}ln\left(\frac{E + p_z}{E - p_z}\right). \tag{111}$$

This corresponds to

$$tanh(y) = \frac{p_z}{E} \tag{112}$$

where, p_z is the longitudinal momentum along the direction of the incident particle, E is the energy, both defined for a given particle. The accessible range of rapidities for a given interaction is determined by the available centre-of-mass energy and all participating particles' rest masses. One usually gives the limit for the incident particle, elastically scattered at zero angle:

$$
\begin{aligned}
|y|_{max} &= ln[(E + p_z)/m] \\
&= ln[\gamma + \gamma\beta] \\
&= ln[\gamma + \sqrt{\gamma^2 - 1}] \\
&\simeq cosh^{-1}\gamma, \quad if \ \gamma \gg 1
\end{aligned}
\tag{113}
$$

with all variables referring to the through-going particle given in the desired frame of reference (e.g. in the centre of mass). A Lorentz boost β along the direction of the incident particle adds a constant, $ln[\gamma + \gamma\beta]$, to the rapidity. Rapidity differences, therefore, are invariant to a Lorentz boost. Statistical particle distributions are flat in y for many physics production models. Note here that $\frac{\partial y}{\partial p_z} = 1/E$.

Furthermore,

$$\gamma = \frac{E_{beam}}{m_p} \tag{114}$$

and for symmetric collision,

$$E_{beam} = \frac{\sqrt{s_{NN}}}{2} \Rightarrow \gamma = \frac{\sqrt{s_{NN}}}{2m_p} \tag{115}$$

Hence,

$$\begin{aligned}
y_{max} &= cosh^{-1}\left[\frac{\sqrt{s_{NN}}}{2m_p}\right] \\
&= cosh^{-1}\left[\frac{E_{lab}}{Am_p}\right] \\
\Rightarrow \boxed{y_{max} = y_b = ln\left[\frac{\sqrt{s_{NN}}}{m_p}\right]}
\end{aligned} \tag{116}$$

Note that the maximum accessible rapidity is independent of the collision species for a symmetric collider and only depends on the center of mass energy.

Examples:
(a) For RHIC top energy, $\sqrt{s_{NN}} = 200$ GeV, $\gamma = E_{beam}/m_p = 106.609$. Hence $y|_{max} = 5.36$. For LHC, center of mass energy of 5.5 TeV ($\gamma = 2931.768$), $y|_{max} = 8.67$.

(b) $y_{beam}^{lab} = 5.8$ for $E_{lab} = 158$ AGeV and 4.4 for $E_{lab} = 40$ AGeV.

M. The Pseudorapidity Variable

Let us assume that a particle is emitted at an angle θ relative to the beam axis. Then its rapidity can be written as
$y = \frac{1}{2} ln\left(\frac{E+P_L}{E-P_L}\right) = \frac{1}{2} ln\left[\frac{\sqrt{m^2+p^2}+p\ cos\ \theta}{\sqrt{m^2+p^2}-p\ cos\ \theta}\right]$. At very high energy, $p \gg m$ and hence

$$\begin{aligned}
y &= \frac{1}{2} ln\left[\frac{p+p\ cos\ \theta}{p-p\ cos\ \theta}\right] \\
&= -ln\ tan\ \theta/2 \equiv \eta
\end{aligned} \tag{117}$$

η is called the pseudorapidity. Hence at very high energy,

$$y \approx \eta = -ln\ tan\ \theta/2. \tag{118}$$

In terms of the momentum, η can be re-written as

$$\eta = \frac{1}{2} ln\left[\frac{|\mathbf{p}|+p_z}{|\mathbf{p}|-p_z}\right]. \tag{119}$$

θ is the only quantity to be measured for the determination of pseudorapidity, independent of any particle identification mechanism. Pseudorapidity is defined for any value of mass,
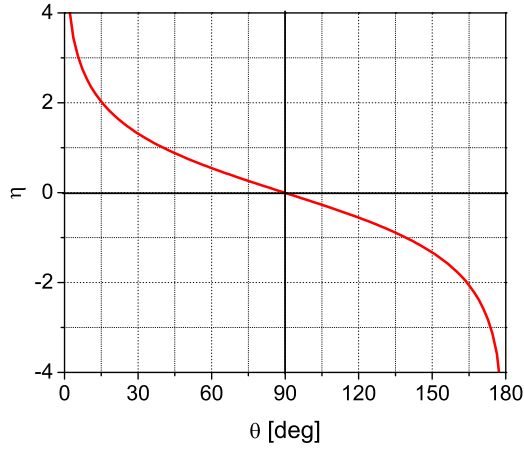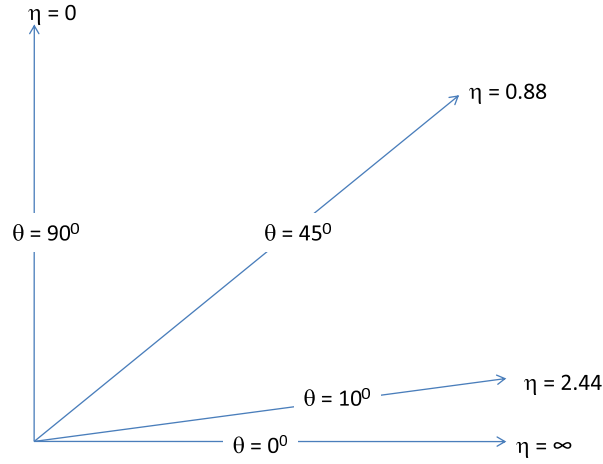
FIG. 9: A plot of pseudorapidity variable, η as a function of the polar angle, θ.

FIG. 10: As angle increases from zero, pseudorapidity decreases from infinity.

momentum and energy of the collision. This also could be measured with or without momentum information which needs a magnetic field. A plot of pseudorapidity as a function of the polar angle, θ is shown in Fig. 9. Table II shows the values of η corresponding to the polar angle of emission a particle. One speaks of the "forward" direction in a collider experiment, which refers to regions of the detector that are close to the beam axis, at high $|\eta|$. The difference in the rapidity of two particles is independent of Lorentz boosts along the beam axis. Pseudorapidity is odd about $\theta = 90$ degrees. In other words, $\eta(\theta) = -\eta(180-\theta)$.

TABLE II: Table of pseudorapidity, η Vs the polar angle, θ.

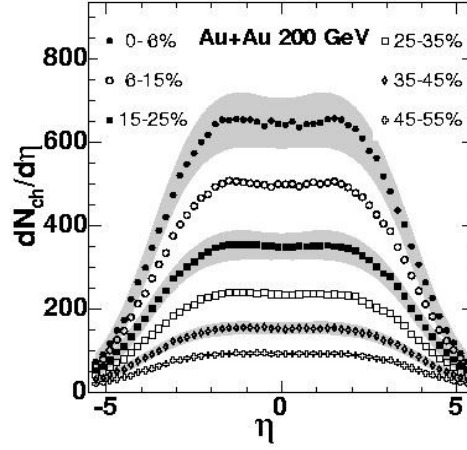| θ	0	5	10	20	30	45	60	80	90	100	110	...	175	180
η	∞	3.13	2.44	1.74	1.32	0.88	0.55	0.175	0	-0.175	-0.55	...	-3.13	$-\infty$

FIG. 11: The mid-rapidity $dN_{ch}/d\eta$ for Au+Au collisions at $\sqrt{s_{NN}} = 200$ GeV [6].

1.   *Change of variables from* $(y, \mathbf{p}_T)$ *to* $(\eta, \mathbf{p}_T)$

By equation 119,

$$e^{\eta} = \sqrt{\frac{|\mathbf{p}| + p_z}{|\mathbf{p}| - p_z}} \tag{120}$$

$$e^{-\eta} = \sqrt{\frac{|\mathbf{p}| - p_z}{|\mathbf{p}| + p_z}} \tag{121}$$

Adding both of the equations, we get

$$|\mathbf{p}| = p_T \, cosh \, \eta \tag{122}$$

$\mathbf{p}_T = \sqrt{|\mathbf{p}|^2 - p_z^2}$. By subtracting the above equations, we get

$$p_z = p_T \, sinh \, \eta \tag{123}$$

Using these equations in the definition of rapidity, we get

$$y = \frac{1}{2} \, ln \left[\frac{\sqrt{p_T^2 \, cosh^2 \, \eta + m^2} + p_T \, sinh \, \eta}{\sqrt{p_T^2 \, cosh^2 \, \eta + m^2} - p_T \, sinh \, \eta} \right] \tag{124}$$

Similarly η could be expressed in terms of y as,

$$\eta = \frac{1}{2} \, ln \left[\frac{\sqrt{m_T^2 \, cosh^2 \, y - m^2} + m_T \, sinh \, y}{\sqrt{m_T^2 \, cosh^2 \, y - m^2} - m_T \, sinh \, y} \right] \tag{125}$$

The distribution of particles as a function of rapidity is related to the distribution as a function of pseudorapidity by the formula

$$\frac{dN}{d\eta d\mathbf{p}_T} = \sqrt{1 - \frac{m^2}{m_T^2 \, cosh^2 \, y}} \, \frac{dN}{dy d\mathbf{p}_T}. \tag{126}$$

In the region $y \gg 0$, the pseudorapidity distribution $(dN/d\eta)$ and the rapidity distribution (dN/dy) which are essentially the $\mathbf{p}_T$-integrated values of $\frac{dN}{d\eta d\mathbf{p}_T}$ and $\frac{dN}{dy d\mathbf{p}_T}$ respectively, are approximately the same. In the region $y \approx 0$, there is a small "*depression*" in $dN/d\eta$ distribution compared to dN/dy distribution due to the above transformation. At very high energies where dN/dy has a mid-rapidity plateau, this transformation gives a small dip in $dN/d\eta$ around $\eta \approx 0$ (see Figure 11). However, for a massless particle like photon, the dip in $dN/d\eta$ is not expected (which is clear from the above equation). Independent of the frame of reference where η is measured, the difference in the maximum magnitude of $dN/d\eta$ appears due to the above transformation. In the CMS, the maximum of the distribution is located at $y \approx \eta \approx 0$ and the η-distribution is suppressed by a factor $\sqrt{1 - m^2/ < m_T^2 >}$ with reference to the rapidity distribution. In the laboratory frame, however, the maximum is located around half of the beam rapidity $\eta \approx y_b/2$ and the suppression factor is $\sqrt{1 - m^2/ < m_T^2 > \cosh^2(y_b/2)}$, which is about unity. Given the fact that the shape of the rapidity distribution is independent of frame of reference, the peak value of the pseudorapidity distribution in the CMS frame is lower than its value in LS. This suppression factor at SPS energies is $\sim 0.8 - 0.9$.

It's to note here again that the conversion of rapidity to pseudorapidity phase space is associated with a Jacobian $J(y, \eta)$, which is given by the right hand side multiplier of Eqn. 126. This depends on the momentum distribution of the produced particles. In the limit of rest mass of the particles being much smaller than their momenta, $J(y, \eta) = 1$. The value of the Jacobian is smaller at LHC energies, compared to that at RHIC energies, as the average transverse momentum of particles increases with beam energy. As measured by the PHENIX experiment, for central Au+Au collisions at $\sqrt{s_{NN}} = 200$ GeV, $J(y, \eta) = 1.25$. Whereas, the corresponding measurement of $J(y, \eta) = 1.09$ for Pb+Pb central collisions at $\sqrt{s_{NN}} = 2.76$ TeV by the CMS experiment at LHC. Rewriting Eqn. 126 after taking an integration over p_T, one obtains:

$$\frac{dN}{d\eta} = v(y)\frac{dN}{dy} \tag{127}$$

where v is the velocity of the particle. For a hadron of mass m and momentum p, which emerges at an angle 90^0 with respect to the beam direction, $y = \eta = 0$, the above relationship becomes,

$$\frac{dN}{d\eta}\Big|_{\eta=0} = v\frac{dN}{dy}\Big|_{y=0} \tag{128}$$

As most of the particles are pions in the final state, with an average momentum of three times the pion mass, we have $v = 0.95$. At mid-rapidity, the rapidity-pseudorapidity conversion hence involves with almost a constant factor. Hence the shape is not affected to a greater extent. However, when one considers the whole rapidity range, where the particle velocity in fact becomes a function of the rapidity, the shape of the pseudorapidity distribution (which is characterized by the width) is different from the rapidity distribution.

Usually, the rapidity spectra are parametrized by the sum of two Gaussian distributions positioned symmetrically with respect to mid-rapidity [7, 8],

$$\frac{dN}{dy} = \frac{\langle N \rangle}{2\sqrt{2\pi\sigma^2}} \left\{ exp\left[-\frac{1}{2}\left(\frac{y - y_0}{\sigma}\right)^2 \right] + exp\left[-\frac{1}{2}\left(\frac{y + y_0}{\sigma}\right)^2 \right] \right\} \tag{129}$$

where $\langle N \rangle$, σ and y_0 are fit parameters. σ is the width of the rapidity distribution.

Landau's energy dependent Gaussian rapidity distribution is given by [9, 10]

$$\frac{1}{\sigma_{in}}\frac{d\sigma}{dy} = \frac{dN}{dy}$$
$$= \frac{N}{(2\pi L)^{1/2}} \, exp. \left(-y^2/2L\right) \tag{130}$$

with parameters

$$L = \frac{1}{2} \, ln \left(s/4m_p^2\right)$$
$$= ln \, \gamma = ln \left(\sqrt{s_{NN}}/2m_p\right) \tag{131}$$

where $s \equiv$ squared total center of mass energy. Comparing Eqns. 129 and 130, we get

$$\boxed{\sigma_y = \sqrt{ln(\sqrt{s_{NN}}/2m_p)}} \tag{132}$$

σ_y is the width of the rapidity distribution, with m_p being the mass of proton. Interestingly, the width of the rapidity distribution is related to the longitudinal flow and velocity of sound in the medium and hence can probe the equation of state of the produced matter. However, this is affected by the final state rescattering, which could be understood through a p_T dependent study of σ_y to disentangle the initial hard scattering from the final state rescattering. With the assumption that the velocity of sound c_s is independent of temperature, the rapidity density in the framework of Landau hydrodynamic model is given by [11, 12]

$$\frac{dN}{dy} = K \, \frac{s_{NN}^{1/4}}{\sqrt{2\pi \, \sigma_y^2}} \, exp. \left(-\frac{y^2}{2\sigma_y^2}\right) \tag{133}$$

where

$$\sigma_y^2 = \frac{8}{3} \, \frac{c_s^2}{1 - c_s^4} \, ln \left(\sqrt{s_{NN}}/2m_p\right)$$
$$\Rightarrow \boxed{\sigma_y^2 = \frac{8}{3} \, \frac{c_s^2}{1 - c_s^4} \, ln \, \gamma} \tag{134}$$

and $K \equiv$ normalization factor. Inverting the above equation for σ_y^2, one can get [13]

$$c_s^2 = \frac{-4}{3} \, \frac{ln \left(\sqrt{s_{NN}}/2m_p\right)}{\sigma_y^2} + \sqrt{\left[\frac{4}{3} \, \frac{ln \left(\sqrt{s_{NN}}/2m_p\right)}{\sigma_y^2}\right]^2 + 1} \tag{135}$$

For an ideal gas (Landau model prediction) the velocity of sound is, $c_s^2 = 1/3$. However, for a gas of hadrons, $c_s^2 = 1/5$. This implies, the expansion of the system is slower compared to an ideal gas scenario. The equation of state is given by
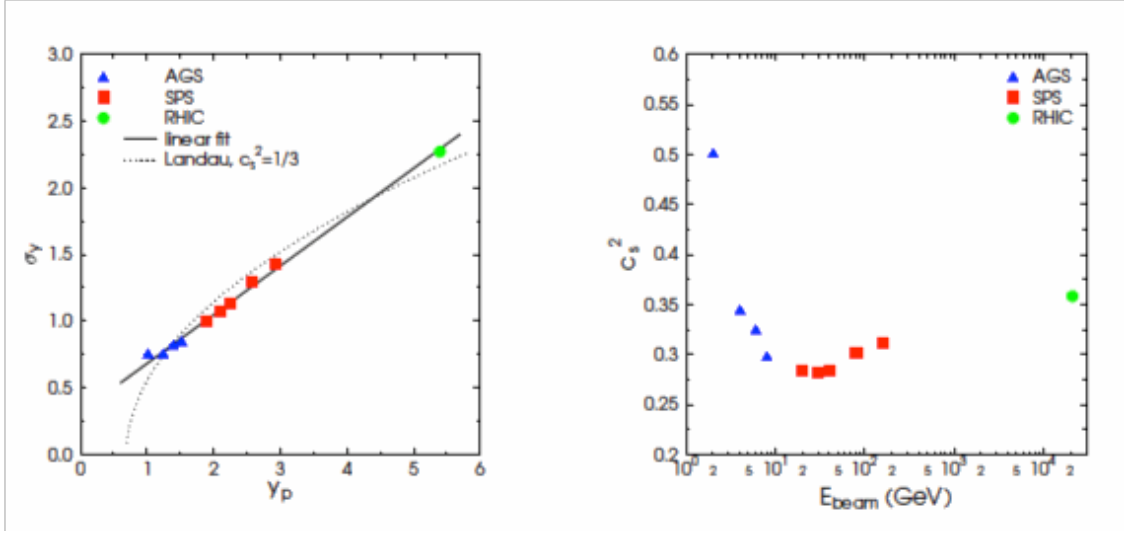
FIG. 12: Left: The width of rapidity distributions of π^- in central Pb+Pb (Au+Au) collisions as a function the beam rapidity. The dotted line indicates Landau model predictions with $c_s^2 = 1/3$, while the full line shows a linear fit to data points. Right: Speed of sound as a function of beam energy showing a softest point at $E_{beam} = 30A$ GeV [13].

$$\boxed{\frac{\partial P}{\partial \epsilon} = c_s^2} \tag{136}$$

where, P is the pressure and ϵ is the energy of the system under consideration. It should be noted here that when the expansion of the matter proceeds as longitudinal and superimposed transverse expansions, a rarefaction wave moves radially inwards with the velocity of sound. The velocity of sound in the medium formed in heavy-ion collisions, when studied as a function of collision (beam) energy, shows the softest point occurring around $E_{beam} = 30$ AGeV. This could be a signature of deconfinement transition [13].

Notation [14]	
Coordinate Space	Momentum Space
$(t, x, y, z) \rightarrow (\tau, r, \phi, \eta)$	$(E, p_x, p_y, p_z) \rightarrow (m_T, \vec{p_T}, y)$
$\tau = \sqrt{t^2 - z^2}$ = proper time	$m_T = \sqrt{E^2 - p_z^2}$ = transverse mass
$\eta = tanh^{-1}\frac{z}{t}$ = space-time rapidity	$y = tanh^{-1}\frac{p_z}{E} = tanh^{-1}v_z$ = rapidity
Conversely,	Conversely,
$t = \tau \ cosh \ \eta, \ z = \tau \ sinh \ \eta$	$E = m_T \ cosh \ y, \ p_z = m_T \ sinh \ y$

N. The Invariant Yield

The rapidity variable has the useful property that it transforms linearly under a Lorentz transformation so that the invariant differential single particle inclusive cross section becomes:

$$\boxed{\frac{Ed^3\sigma}{dp^3} = \frac{Ed^3\sigma}{p_T dp_T dp_L d\phi} = \frac{d^3\sigma}{p_T dp_T dy d\phi}}, \tag{137}$$

where

$$dy = \frac{dp_L}{E} \tag{138}$$

First we proceed to show $\frac{d^3p}{E}$ is Lorentz invariant. The differential of Lorentz boost in longitudinal direction is given by

$$dp_z^* = \gamma(dp_z - \beta dE). \tag{139}$$

Taking the derivative of the equation $E^2 = p^2 + m^2$, we get

$$EdE = p_z dp_z. \tag{140}$$

Using equations 139 and 140 we get

$$dp_z^* = \gamma(dp_z - \beta\frac{p_z dp_z}{E})$$

$$= \frac{dp_z}{E} \ E^*. \tag{141}$$

As $\mathbf{p}_T$ is Lorentz invariant, multiplying $\mathbf{p}_T$ on both the sides and re-arranging gives

$$\frac{d^3p^*}{E^*} = \frac{d^2\mathbf{p}_T \ dp_z}{E} = \frac{d^3p}{E}. \tag{142}$$

In terms of experimentally measurable quantities, $\frac{d^3p}{E}$ could be expressed as

$$\frac{d^3p}{E} = d\mathbf{p}_T \ dy$$

$$= p_T dp_T d\phi dy \tag{143}$$

$$= m_T dm_T d\phi dy. \tag{144}$$

The Lorentz invariant differential cross-section $\frac{Ed^3\sigma}{dp^3} = \frac{Ed^3N}{dp^3}$ is the invariant yield. In terms of experimentally measurable quantities this could be expressed as

$$\frac{Ed^3\sigma}{dp^3} = \frac{1}{m_T}\frac{d^3N}{dm_Td\phi dy}$$
$$= \frac{1}{2\pi\, m_T}\frac{d^2N}{dm_Tdy}$$
$$= \frac{1}{2\pi\, p_T}\frac{d^2N}{dp_Tdy}. \tag{145}$$

To measure the invariant yields of identified particles equation 145 is used experimentally.

O. Inclusive Production of Particles and the Feynman Scaling variable x_F

A reaction of type
beam + target $\longrightarrow$ A + anything
where A is known is called an "*inclusive reaction*". The cross-section for particle production could be written separately as functions of $\mathbf{p}_T$ and p_L:

$$\sigma = f(\mathbf{p}_T)g(p_L). \tag{146}$$

This factorization is empirical and convenient because each of these factors has simple parametrizations which fit well to experimental data.

Similarly the differential cross-section could be expressed by

$$\frac{d^3\sigma}{dp^3} = \frac{d^2\sigma}{\mathbf{p}_T^2}\frac{d\sigma}{dp_L} \tag{147}$$

Define the variable

$$x_F = \frac{p_L^*}{p_L^*(max)}$$
$$\Rightarrow \boxed{x_F = \frac{2p_L^*}{\sqrt{s}}} \tag{148}$$

x_F is called the *Feynman scaling variable*: longitudinal component of the cross-section when measured in CMS of the collision, would scale *i.e.* would not depend on the energy $\sqrt{s}$. This is the fraction of maximum allowed longitudinal momentum ($-1 \leq x_F \leq 1$) carried by the particle in the CMS. This is used in comparing the shapes of particle distributions at different collision energies near the projectile or target rapidity. Instead of $\frac{d\sigma}{dp_L^*}$, $\frac{d\sigma}{dx_F}$ is measured which wouldn't depend on energy of the reaction, $\sqrt{s}$. This Feynman's assumption is valid approximately. This variable is usually used to compare particle distribution at different collision energies.

The differential cross-section for the inclusive production of a particle is then written as

$$\frac{d^3\sigma}{dx_Fd^2\mathbf{p}_T} = F(s,\ x_F,\ \mathbf{p}_T) \tag{149}$$

Feynman's assumption that at high energies the function $F(s,\ x_F,\ \mathbf{p}_T)$ becomes asymptotically independent of the energy means:

$lim_{s \to \infty} F(s,\ x_F,\ \mathbf{p}_T)\ =\ F(x_F,\ \mathbf{p}_T)\ =\ f(\mathbf{p}_T)\ g(x_F)$

According to Feynman, the mean number of particles rises logarithmically in the asymptotic limit of large energies. This in fact applies to any kind of particles:

$< N >\ \ \propto\ ln\ W\ \propto ln\ \sqrt{s},$

where $W = \sqrt{s}/2$, is the beam energy of a symmetric collider. However, these conclusions are based on phenomenological arguments about the exchange of quantum numbers between the colliding particles. It was assumed that the number of particles with a given mass and transverse momentum $(\mathbf{p}_T)$ in a longitudinal interval p_z depends on the energy $E\ =\ E(p_Z)$ as

$$\frac{dN}{dp_z} \sim \frac{1}{E} \tag{150}$$

This means the among all the produced particles, the probability of finding a particle of kind i with transverse momentum $\mathbf{p}_T$, mass m and longitudinal momentum p_z is of the form:

$$f_i(\mathbf{p}_T,\ x_F = p_z/W)\ \frac{dp_z}{E}\ d^2p_T \tag{151}$$

with $E\ =\ \sqrt{m^2 + p_T^2 + p_z^2}$ being the total energy of the particle under discussion. The function $f_i(\mathbf{p}_T,\ x_F)$ denotes the particle distribution. Let's derive Eqn. 151 to have a hand on experience.

Rewriting Eqn. 151 in the form of invariant cross section

$$\frac{1}{\sigma}\ E\frac{d^3\sigma}{dp_z\ d^2\mathbf{p}_T}\ =\ f_i(\mathbf{p}_T,\ x_F) \tag{152}$$

f_i factorizes approximately (found experimentally) and a normalization g_i is chosen such that

$$\int f_i(\mathbf{p}_T,\ x_F)\ d^2p_T\ =\ f_i(x_F) \int g_i(\mathbf{p}_T)\ d^2p_T$$
$$=\ f(x_F) \tag{153}$$

Here

$$\int g_i(\mathbf{p}_T)\ d^2p_T\ =\ 1 \tag{154}$$

Now, integrating Eqn. 152 and applying Eqn. 153, we get

$$\int \frac{1}{\sigma}\ E\ \frac{d^3\sigma}{dp_z\ d^2\mathbf{p}_T}\ \frac{d^3p}{E}\ =\ < N >$$
$$=\ \int f_i(\mathbf{p}_T,\ x_F)\ \frac{d^3p}{E}$$
$$=\ \int f_i(x_F)\frac{dp_z}{\sqrt{W^2x_F^2 + m_T^2}} \tag{155}$$

On the left hand side we have used the definition of invariant cross section with the average particle multiplicity $< N >$, and for m_{T} an effective average p_{T} is used with $x_{\mathrm{F}} = p_{\mathrm{z}}/W$. Hence

$$< N > = \int_{-1}^{+1} f_i(x_{\mathrm{F}}) \frac{\mathrm{d}x_F}{\sqrt{x_F^2 + \frac{m_T^2}{W^2}}}, \qquad (156)$$

with $dx_{\mathrm{F}} = \frac{dp_z}{W}$.

The integral is symmetric because $f_i(x_{\mathrm{F}})$ is symmetric for collisions of identified particles. For asymmetric collision systems, the integration could be performed separately for negative and positive x_{F} and yields the same result.

$f_i(x_{\mathrm{F}}) \leq B$ is finite and bounded due to energy conservation. Feynman assumed that for $x_{\mathrm{F}} = 0$ a finite limit is reached.

$$x_{\mathrm{F}} \to 0 \quad \Rightarrow \quad \frac{p_z}{W} = \frac{p_z}{E_{lab}} \quad \to \quad 0$$
$$\Rightarrow E_{lab} \to \infty$$

Hence

$$< N > = 2 \int_0^1 f_i(x_{\mathrm{F}}) \frac{\mathrm{d}x_F}{\sqrt{x_F^2 + \frac{m_T^2}{W^2}}} \leq 2 \int_0^1 B \frac{\mathrm{d}x_F}{\sqrt{x_F^2 + \frac{m_T^2}{W^2}}}$$

$$= 2B \, ln \left[x_F + \sqrt{x_F^2 + \frac{m_T^2}{W^2}} \right] \Big|_0^1$$

$$= 2B \, ln \left[1 + \sqrt{1 + \frac{m_T^2}{W^2}} \right] - 2B \, ln \left(\frac{m_T}{W} \right) \qquad (157)$$

In the limit $W \to \infty$, the first term of the above equation could be shown to be constant and the second term is proportional to $ln \, W$. hence, Feynman scaling tells that the average total multiplicity scales as:

$$\boxed{< N > \propto ln \, W \propto ln \, \sqrt{s}} \qquad (158)$$

If one considers the maximum reachable rapidity in a collision to increase like $ln \, \sqrt{s}$ ($y_{\max} \sim ln \, \sqrt{s}$) and in addition, particles are evenly distributed in rapidity, it follows that $\frac{dN}{dy}$ is independent of $\sqrt{s}$:

$$\boxed{\frac{dN}{dy} = Constant} \qquad (159)$$

Feynman's Assumption: $f_i(p_{\mathrm{T}}, x_{\mathrm{F}})$ which denotes the particle distribution, becomes independent of W at high energies. This assumption is known as Feynman Scaling and f_i

is called the scaling function or Feynman function. The variable

$$x_F \;=\; p_z/W \;=\; 2p_z/\sqrt{s} \tag{160}$$

is called *Feynman-x.*

Feynman-x is the ratio of the longitudinal momentum of the particle to the total energy of the incident particle. **Note:**

1. $\frac{dN}{dy}$ = Constant $\Rightarrow$ Height of the rapidity distribution around mid-rapidity (the so-called plateau) is independent of collision energy, $\sqrt{s}$.

2. Equivalently, if Feynman scaling holds good, the pseudorapidity density of charged particles at mid-rapidity *i.e.* $\frac{dN}{d\eta}(\eta = 0)$ is approximately constant.

Scaling properties of various distributions can also be studied in terms of the scaled rapidity:

$$z = y^*/y^*_{\max}. \tag{161}$$

The two scaled variables: x_F and z emphasize different kinematic regions: the detailed structure of the central part of the distribution (*i.e.* large emission angles) can be better seen using x_F, while the far *"wings"* (*i.e.* small angles) using z.

P.   What is a $\mathbf{p}_T$ or m_T Spectrum?

$\mathbf{p}_T$ spectrum: $\frac{1}{2\pi p_T}\frac{d^2N}{dy\,dp_T}$ Vs p_T

m_T spectrum: $\frac{1}{2\pi m_T}\frac{d^2N}{dy\,dm_T}$ Vs m_T or $(m_T - m_0)$

The invariant yield when plotted as a function of $\mathbf{p}_T$ or m_T or $(m_T - m_0)$ is called $\mathbf{p}_T$ or m_T spectrum, respectively. Experimentally at kinetic freeze-out when the elastic collisions between the final state particles almost cease to happen (in other words the particle mean free path becomes higher than the system size: size of the produced fireball) then the $\mathbf{p}_T$ or m_T-spectrum is frozen, which carries the kinetic freeze-out properties of the system. Recall here that $\boxed{p_T dp_T \;=\; m_T dm_T}$.

The distribution of particles as a function of $\mathbf{p}_T$ is called $\mathbf{p}_T$-distribution. Mathematically,

$$\frac{dN}{d\mathbf{p}_T} \;=\; \frac{dN}{2\pi\,|\mathbf{p}_T|d|\mathbf{p}_T|} \tag{162}$$

where dN is the number of particles in a particular $\mathbf{p}_T$-bin. People usually plot $\frac{dN}{p_T dp_T}$ as a function of p_T taking out the factor $1/2\pi$ which is a constant. Here p_T is a scalar quantity. The low-p_T part of the p_T-spectrum is well described by an exponential function having thermal origin given by Eqn. (163). However, a QCD-inspired power-law function (given by Eqn. (164)) seems to provide a better description of the high p_T ($\gtrsim 3$ GeV/c) region. To describe the whole range of the p_T-spectrum, one uses the Levy function given by Eqn.(165)

which has an exponential part to describe low-p_T and a power-law function to describe the hight-p_T part which is dominated by hard scatterings (high momentum transfer at early times of the collision).

$$\frac{1}{2\pi p_T}\frac{d^2N}{dy\,dp_T} \;=\; A\,e^{\frac{-m_T}{T}}, \tag{163}$$

$$\frac{1}{2\pi p_T}\frac{d^2N}{dy\,dp_T} \;=\; B\left(1+\frac{p_T}{p_0}\right)^{-n}, \tag{164}$$

$$\frac{1}{2\pi p_T}\frac{d^2N}{dy\,dp_T} \;=\; \frac{dN}{dy}\frac{(n-1)(n-2)}{2\pi nC[nC+m_0(n-2)]}\times\left(1+\frac{\sqrt{p_T^2+m_0^2}-m_0}{nC}\right)^{-n}, \tag{165}$$

where $A, T, B, p_0, n, \frac{dN}{dy}, C$, and m_0 are fit parameters [15]. The inverse slope parameter of p_T-spectra is called the effective temperature (T_{eff}), which has a thermal contribution because of the random kinetic motion of the produced particles and a contribution from the collective motion of the particles. This will be described in details in the section of freeze-out properties and how to determine the chemical and kinetic freeze-out temperatures experimentally.

If we look deeper into Eqn. 163, we expect this for a 2-dimensional classical thermalized fluid at rest. We recall here that Boltzmann distribution $\propto exp.(-\beta E)$, where $\beta \equiv 1/T$ is the inverse temperature. We use a semi-logarithmic plot (y-axis logarithmic scale) of $N\,exp.\,(-\beta m_T)$ vs $(m_T - m_0)$ which happens to be a straight line with slope $-\beta$, from which the effective temperature is extracted. A semi-log plot of $N\,exp.\,(-\beta m_T)$ vs p_T is an approximate straight line if $p_T \gg m_0$.

The most important parameter is then the mean p_T which carries the information of the effective temperature of the system. Experimentally, $\langle p_T \rangle$ is studied as a function of $\frac{dN_{ch}}{d\eta}$ which is the measure of the entropy density of the system. This is like studying the temperature as a function of entropy to see the signal of phase transition. The phase transition is of 1st order if a plateau is observed in the spectrum signalling the existence of latent heat of the system (like liquid-vapour phase transition). This was first proposed by L. Van Hove [16].

The average of any quantity A following a particular probability distribution $f(A)$ can be written as

$$\langle A \rangle \;=\; \frac{\int A\,f(A)\,dA}{\int f(A)\,dA}. \tag{166}$$

Similarly,

$$\begin{aligned}\langle p_T \rangle \;&=\; \frac{\int_0^\infty p_T\,(\frac{dN}{dp_T})\,dp_T}{\int_0^\infty(\frac{dN}{dp_T})\,dp_T}\\[2mm] &=\; \frac{\int_0^\infty p_T\,dp_T\;\;p_T(\frac{dN}{p_T dp_T})}{\int_0^\infty p_T\,dp_T(\frac{dN}{p_T dp_T})}\\[2mm] &=\; \frac{\int_0^\infty p_T\,dp_T\;\;p_T\,f(p_T)}{\int_0^\infty p_T\,dp_T f(p_T)}\end{aligned} \tag{167}$$

where $2\pi \ p_T \ dp_T$ is the phase space factor and the p_T-distribution function is given by

$$f(p_T) \ = \ \frac{dN}{d\mathbf{p}_T} \ = \ \frac{dN}{p_T dp_T}. \tag{168}$$

Example Experimental data on p_T-spectra are sometimes fitted to the exponential Boltzmann type function given by

$$f(p_T) \ = \ \frac{1}{p_T} \frac{dN}{dp_T} \ \simeq \ C \ e^{-m_T/T_{eff}}. \tag{169}$$

The $\langle m_T \rangle$ could be obtained by

$$\begin{aligned}
\langle m_T \rangle \ &= \ \frac{\int_0^\infty p_T \ dp_T \ m_T \ exp.(-m_T/T_{eff})}{\int_0^\infty p_T \ dp_T \ exp.(-m_T/T_{eff})} \\
&= \ \frac{2T_{eff}^2 + 2m_0 T_{eff} + m_0^2}{m_0 + T_{eff}}
\end{aligned} \tag{170}$$

$$\Rightarrow \boxed{\langle m_T \rangle = T_{eff} + m_0 + \frac{(T_{eff})^2}{m_0 + T_{eff}}} \tag{171}$$

where m_0 is the rest mass of the particle. It can be seen from the above expression that for a massless particle

$$\langle m_T \rangle \ = \ \langle p_T \rangle \ = \ 2T_{eff}. \tag{172}$$

This also satisfies the principle of equipartition of energy which is expected for a massless Boltzmann gas in equilibrium. However, in experiments the lower (higher) limit of p_T is a finite quantity. In that case the integration will involve an incomplete gamma function.

Q. How is the radial flow measured from p_T-spectra?

In central heavy-ion collisions, radial flow is supposed to play a vital role in the thermodynamic expansion of the produced fireball. Radial flow is related to the initial pressure produced just after the collision. This could be extracted from the analysis of the transverse momentum spectra. Assuming a thermalized non-relativistic plasma (for simplicity), particle velocity [14]

$$\vec{v} \ = \ \vec{v}_{flow} \ + \ \vec{v}_{th}, \tag{173}$$

where $\vec{v}_{flow} \ \equiv$ transverse velocity of the expanding fluid, which is independent of the particle species and is the collective component of $\vec{v}$.

$\vec{v}_{flow} \ \equiv$ thermal component of $\vec{v}$, which is generated due to random thermal motion of the quanta of the system.

Hence for a particle of mass m_0

$$\begin{aligned}
\left\langle \frac{1}{2} m_0 v^2 \right\rangle \ &= \ \frac{1}{2} m_0 v_{flow}^2 \ + \ \left\langle \frac{1}{2} m_0 v_{th}^2 \right\rangle \\
&= \ \frac{1}{2} m_0 v_{flow}^2 \ + \ \frac{3}{2} kT,
\end{aligned} \tag{174}$$

where T is the temperature of the fluid. Hence the average kinetic energy (K.E) is given by

$$\langle K.E. \rangle = \frac{1}{2} m_0 v_{flow}^2 + \frac{3}{2} kT$$

$$\Rightarrow \frac{3}{2} kT_{eff} = \frac{1}{2} m_0 v_{flow}^2 + \frac{3}{2} kT_{th}$$

$$\Rightarrow T_{eff} = T_{th} + \frac{1}{3} m_0 v_{flow}^2 \ (taking \ k = 1) \tag{175}$$

Because the final state particle are measured at freeze out (after they stream out to reach the detectors), the extracted values of $\vec{v}_{flow}$ and T correspond to the instant of freeze out.

Taking $k = 1$, in 2-dimension:

$$\boxed{T_{eff} = T_{th} + \frac{1}{2} m_0 v_{flow}^2}, \tag{176}$$

and in 1-dimension:

$$\boxed{T_{eff} = T_{th} + m_0 v_{flow}^2} \tag{177}$$

Note that these formulae are used essentially when the spectra is well described by an exponential function (low-p_T regime). However, when one goes to high-p_T regime, the following formula could be used for extracting the radial flow from the p_T or m_T-spectra.

$$\boxed{T_{eff} = T_{th} \sqrt{\frac{1 + v_{flow}}{1 - v_{flow}}}} \tag{178}$$

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